THE SOLUTION OF EQUATIONS BY CONTINUED FRACTIONS

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1. Introduction. Continued fractions may be used to define rapidly convergent methods for solving equations. We shall study in this paper a set of continued fraction approximants P_k/Q_k for correcting a first estimate x to a required root ξ of any given equation $\phi(x)=0$, provided that $\phi(x)$ has as many continuous derivatives as may be required in a neighborhood U of the root ξ. We write

$$(1.1)\qquad y=\phi(x),\quad y'=\phi'(x),\quad y''=\phi''(x),\cdots;\quad \phi(\xi)=0.$$

$$(1.2)\qquad \nu=-y/y',\quad \beta=y''/2y',\quad \gamma=y'''/3y'',\quad \delta=y^{iv}/4y''',\cdots;$$

and we note that the function ν is the familiar Newton-Raphson estimate for the exact correction $\epsilon=\xi-x$ that is to be added to the initial value x.*

Upon expanding the correction ϵ in a continued fraction with partial numerators a_k, denominators 1, and remainders r_k, we write

$$(1.3)\qquad \epsilon=\xi-x=\cfrac{a_1}{1+\cfrac{a_2}{1+\cdots\cfrac{}{1+\cfrac{a_k}{1+r_{k+1}}}}}=\frac{P_k+r_{k+1}P_{k-1}}{Q_k+r_{k+1}Q_{k-1}},$$

$$\text{where}\quad r_k=\frac{a_k}{1+r_{k+1}}.$$

The partial numerators a_k in the continued fraction (1.3) may be determined by means of the series expansion of $\epsilon=r_1=\sum c_k\nu^k$ as a power series in ν (see 3.3) in which c_k is a certain rational function (3.2) of the first $k-1$ of the quantities $\beta, \gamma, \delta, \cdots$, evaluated at the chosen first estimate x. Since the first term in the series for r_1 is $c_1\nu=\nu$, we define $a_1=\nu$, and then by division obtain a second power series for $r_2=(a_1-r_1)/r_1=\beta\nu+\cdots$, which vanishes for $\nu=0$. In general, r_{k+1} will vanish for $\nu=0$ if we define a_k to be the first term in the series expansion for r_k. Assuming that our initial estimate x is not taken at a double root of any of the first k remainders, the ratio a_k/ν will be a rational function of the first $k-1$ quantities $\beta, \gamma, \delta, \cdots$, independent of ν. Thus we may compute successively the first few partial numerators as follows:

$$(1.4)\qquad a_1=\nu,\quad a_2=\nu\beta,\quad a_3=\nu\beta-\nu\gamma,\quad a_4=\nu\beta-\nu\gamma(\gamma-\delta)/(\beta-\gamma).$$

The procedure just outlined is complicated to apply for $k\geqq 4$, but in equation (3.8) of Theorem 2 we shall obtain concise expressions for a_k and r_k as ratios

* In case ξ is a multiple root of $\phi(x)=0$, we replace the given function $\phi(x)$ by the related function $\phi(x)/\phi'(x)$ that has a simple root at $x=\xi$. Thus the function y and the neighborhood U are chosen so that $y'\neq 0$ in U. If one or more of the higher derivatives y'', y''', *etc.*, vanish identically, an appropriate modification can be made to avoid using γ, δ, *etc.* If not, we may assume that U is chosen small enough so that the required derivatives do not vanish between x and ξ.

of certain determinants. However, the following inductive formula based on (4.3) gives a more practical procedure for actually computing consecutively the ratios a_k/ν:

$$(1.5) \qquad (k+1)\frac{a_{k+1}}{\nu} = 2\frac{a_2}{\nu} + (k-2)\frac{a_{k-1}}{\nu} - \frac{a_k}{\nu} - \frac{d}{dx}\ln\left(\frac{a_k}{\nu}\right) \qquad k \geqq 2.$$

The kth approximant to the error $\epsilon = \xi - x$ is defined to be that rational function P_k/Q_k of $a_1, \cdots, a_k$ obtained from (1.3) by replacing r_{k+1} by 0. It will be shown in Theorem 1 of Section 2 that the residual error $\epsilon_k = \epsilon - P_k/Q_k$ is of the order ν^{k+1}.

Whereas the first approximant P_1/Q_1 is the Newton-Raphson correction ν, [2, 3], the second approximant

$$(1.6) \qquad \frac{P_2}{Q_2} = \frac{a_1}{1+a_2} = \frac{\nu}{1+\nu\beta} = \nu - \frac{\nu^2\beta}{1+\nu\beta}$$

defines the more accurate Halley correction [1] which was discovered independently by Frame [4, 5], Wall [16], and others. The third approximant

$$(1.7) \qquad \frac{P_3}{Q_3} = \frac{\nu}{1+\dfrac{\nu\beta}{1+\nu(\beta-\gamma)}} = \nu - \frac{\nu^2\beta}{1+\nu(2\beta-\gamma)}$$

reduces to the correction proposed by J. K. Stewart [13], if the effect of the third derivative is neglected by setting $\gamma = 0$, but is better than Stewart's correction if $\gamma \neq 0$.

How these and higher order approximations compare in some numerical cases will be illustrated in section 2, after a simple expression R_{k+1}/Q_k has been obtained for the residual error $\epsilon - P_k/Q_k$ and somewhat similar expressions $A_k/Q_{k-1}Q_k$ and $A_k/Q_{k-1}Q_{k+1}$ for the differences of successive and alternate approximants. (Here $R_k = (-1)^{k-1}r_1r_2\cdots r_k$, and $A_k = (-1)^{k-1}a_1a_2\cdots a_k$).

Explicit expressions for the r_k and a_k will be obtained as ratios of determinants in Theorem 2 of Section 3.

Five derivative relations in Lemma 2 of Section 4, such as the one from which (1.5) was derived, furnish the basis for an independent proof of the formula of Tauber expressing the derivative of the iteration function $f_k(x) = x + P_k/Q_k$ in the form

$$(1.8) \qquad f_k'(x) = (x + P_k/Q_k)' = (k+1)(-1)^k a_2a_3\cdots a_{k+1}/Q_k^2.$$

Theorem 3 also gives bounds and estimates for the residual error.

Theorem 4 of Section 5 treats the case of positive numerators $a_2, \cdots, a_{k+1}$ and shows that in this case the function $f_k(x)$ gives a better approximation to ϵ than is obtained from the kth partial sum of the corresponding power series. Here then is some solid ground for preferring the continued fraction method

among the many iteration methods that have been considered.

Several of the many papers that have discussed methods of solving equations are listed in the Index to volumes 1 to 56 of this MONTHLY under "Approximate Methods of Solution." Hamilton [8] and Householder [9] have recent papers on the subject, and Householder refers to a fundamental paper of E. Schroeder [12] in 1870 in which iterative methods of solution are discussed in general terms.

In the general iterative method for approximating a root ξ of $\phi(x)=0$, an iteration function $f(x)$ is constructed, such that if $x=x_1$ is chosen arbitrarily in some sufficiently small neighborhood U of the root ξ, then the sequence $\{x_n\}$ defined by

$$x_{n+1} = f(x_n) \tag{1.9}$$

converges to the root ξ. If k is the largest positive number such that

$$\lim_{x\to\xi} \frac{|\xi - f(x)|}{|\xi - x|^k} < \infty, \tag{1.10}$$

then the convergence to ξ is said to be of order k.

Sufficient conditions on $f(x)$ to insure convergence in the neighborhood U were given by Schroeder [12] as follows:

(1.11a) $f(\xi) = \xi$ for some ξ in U;

(1.11b) $f(x)$ and $f'(x)$ continuous in U;

(1.11c) $|f'(x)| \leqq \theta$ for fixed $\theta < 1$ and all x in U.

Thus a region of convergence of the iteration functions

$$f_k(x) = x + P_k/Q_k \tag{1.12}$$

is indicated by the bound on $|f_k'(x)|$ in Theorem 3, and the convergence is seen to be of order $k+1$.

2. The continued fraction approximants. Having selected an arbitrary initial estimate x in a sufficiently small neighborhood U of the root ξ of the equation $\phi(x)=0$, we denote the exact difference $\epsilon=\xi-x$ by r_1, and denote the correction ν of Newton's method by a_1. Then as above we define successive remainders r_k recursively by the equations

$$r_1 = \epsilon = \xi - x, \qquad r_{k+1} = (a_k/r_k) - 1, \tag{2.1}$$

where the partial numerators a_k satisfy the conditions that a_k/a_1 is a rational function of the first $k-1$ of the quantities $\beta, \gamma, \delta, \cdots$, that approximates r_k/ν in the sense that

$$\lim_{\nu\to 0} (a_k/r_k) = 1. \tag{2.2}$$

The numerator P_k and denominator Q_k of the kth approximant P_k/Q_k in (1.3)

are polynomials in $a_1, a_2, \cdots, a_k$ without common factors, that may be defined either by the recurrence relations

$$(2.3)\qquad \begin{aligned} P_0 &= 0, \quad P_1 = a_1, \qquad P_k = P_{k-1} + a_k P_{k-2} \\ Q_0 &= 1, \quad Q_1 = 1, \qquad Q_k = Q_{k-1} + a_k Q_{k-2} \end{aligned}$$

or by the equivalent matrix equation [6]:

$$(2.4)\qquad \begin{pmatrix} P_{k-1} & P_k \\ Q_{k-1} & Q_k \end{pmatrix} = \begin{pmatrix} 0 & a_1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & a_2 \\ 1 & 1 \end{pmatrix}\cdots\begin{pmatrix} 0 & a_k \\ 1 & 1 \end{pmatrix}.$$

The determinant $-A_k$ of the left member of (2.4) is the product of the k determinants $-a_j$ on the right, so by (2.3) and (2.4) we have

$$(2.5)\qquad A_k = \begin{vmatrix} P_k & P_{k-1} \\ Q_k & Q_{k-1} \end{vmatrix} = \begin{vmatrix} P_{k+1} & P_{k-1} \\ Q_{k+1} & Q_{k-1} \end{vmatrix} = (-1)^{k-1}a_1a_2\cdots a_k;$$

$$A_0 = -1, \qquad A_1 = a_1 = \nu.$$

The corresponding products R_k of the remainders r_k are defined as follows:

$$(2.6)\quad R_k = (-1)^{k-1}r_1r_2\cdots r_k;\ R_0 = -1;\ R_1 = r_1 = \epsilon;\ R_2 = -r_1r_2 = \epsilon - \nu, \textit{ etc.}$$

THEOREM 1. *The residual error of the kth approximant P_k/Q_k to the continued fraction* (1.2) *is given by the equation*

$$(2.7)\qquad \epsilon_k \equiv \epsilon - \frac{P_k}{Q_k} = \frac{R_{k+1}}{Q_k}.$$

It is nearly equal to A_{k+1} and is of order $k+1$ in ν when $|\nu|$ is small.

Proof: From equations (2.6) and (2.1) we have

$$(2.8)\qquad R_{k+1} = r_{k+1}r_kR_{k-1} = (a_k - r_k)R_{k-1} = R_k + a_kR_{k-1}.$$

Hence R_{k+1} satisfies the same recurrence relation as do P_k and Q_k in (2.3). Thus

$$(2.9)\qquad (R_{k+1} + P_k - \epsilon Q_k) = (R_k + P_{k-1} - \epsilon Q_{k-1}) + a_k(R_{k-1} + P_{k-2} - \epsilon Q_{k-2}).$$

Since by direct substitution we have

$$(2.10)\quad R_1 + P_0 - \epsilon Q_0 = \epsilon + 0 - \epsilon = 0;\ R_2 + P_1 - \epsilon Q_1 = (\epsilon - \nu) + \nu - \epsilon = 0,$$

it follows by induction that the left member of (2.9) vanishes for all $k>0$. This proves equation (2.7).

Using (2.7) and (2.5) we may express the denominator of the right member of (1.3) as follows:

$$(2.11)\quad Q_k + r_{k+1}Q_{k-1} = \frac{Q_kQ_{k-1}}{R_k}\left(\frac{R_k}{Q_{k-1}} - \frac{R_{k+1}}{Q_k}\right) = \frac{Q_kQ_{k-1}}{R_k}\left(\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}}\right) = \frac{A_k}{R_k}.$$

The ratio A_k/R_k is nearly 1 when ν is small, since it is the product of ratios

a_k/r_k which approach 1, by (2.2). Hence by (2.11) and (2.6) Q_k is near to 1, and both R_{k+1} and R_{k+1}/Q_k vanish to the order $k+1$ in ν.

The ratio A_k/Q_kQ_{k-1} (or the ratio $A_k/Q_{k+1}Q_{k-1}$) is seen from (2.5) to be the difference of successive (or alternating) approximants P_k/Q_k. Hence the exact correction ϵ may be written as a series with remainder in each of the three following ways:

$$\epsilon = \frac{A_1}{Q_1} + \frac{A_2}{Q_1Q_2} + \frac{A_3}{Q_2Q_3} + \cdots + \frac{A_k}{Q_{k-1}Q_k} + \frac{R_{k+1}}{Q_k}, \tag{2.12}$$

$$\epsilon = \frac{A_1}{Q_1} + \frac{A_2}{Q_1Q_3} + \frac{A_4}{Q_3Q_5} + \cdots + \frac{A_{2k-2}}{Q_{2k-3}Q_{2k-1}} + \frac{R_{2k}}{Q_{2k-1}}, \tag{2.13}$$

$$\epsilon = \frac{A_1}{Q_2} + \frac{A_3}{Q_2Q_4} + \frac{A_5}{Q_4Q_6} + \cdots + \frac{A_{2k-1}}{Q_{2k-2}Q_{2k}} + \frac{R_{2k+1}}{Q_{2k}}. \tag{2.14}$$

Later we shall obtain general expressions for the partial numerators a_k, but let us first see in some particular examples how much the right members of (2.12), (2.13), (2.14) improve upon Newton's correction, when the remainders R_3, R_4, and R_5 are neglected.

Consider first the equation

$$y = e^z - 1 - c = 0, \qquad |c| \leqq 1, \tag{2.15}$$

and take $x = x_1 = 0$ as the initial approximation. The solution by series would be the familiar logarithmic expansion

$$\xi = \ln(1+c) = c - \frac{c^2}{2} + \frac{c^3}{3} - \frac{c^4}{4} + \cdots \tag{2.16}$$

which converges very slowly when $c = 1$. By the continued fraction method of solution, we have at $x = x_1 = 0$,

$$\begin{aligned} &y = -c, \quad y' = 1, \quad y'' = 1, \quad y''' = 1, \quad y^{iv} = 1, \quad \textit{etc.}, \\ &\nu = c, \quad \beta = 1/2, \quad \gamma = 1/3, \quad \delta = 1/4, \cdots, \\ &a_1 = c, \quad a_2 = c/2, \quad a_3 = c/6, \quad a_4 = c/3, \text{ and in general} \\ &\qquad a_{2k} = \frac{2k}{2k-1}\,\frac{c}{4}, \qquad a_{2k+1} = \frac{2k}{2k+1}\,\frac{c}{4}. \end{aligned} \tag{2.17}$$

Thus the fourth approximant for $\ln(1+c)$ is

$$\ln(1+c) \sim \cfrac{c}{1 + \cfrac{c/2}{1 + \cfrac{c/6}{1 + c/3}}} = \frac{c + c^2/2}{1 + c + c^2/6}; \quad \ln z \sim \frac{3(z^2-1)}{z^2 + 4z + 1}. \tag{2.18}$$

For $c=1$, the successive approximants for the value $\ln 2=0.69315$ are:

(2.19) 1.000 (Newton); 0.667 (Halley); 0.7000 (P_3/Q_3); 0.69231 (P_4/Q_4).

For values of c less than 1 but positive, the residual error of P_4/Q_4 is approximately $c^5/1200$, whereas in the corresponding Taylor series (2.16) the fourth partial sum (7/12) has the much larger residual error of about 0.110 for $c=1$, or $c^5/(5+4c)$ for $0<c<1$.

For a second example, we shall calculate the root near $x_1=2$ of the equation $y=x^3-3x-1=0$. We compute successively the columns below:

Convergents to root $\xi=2+\epsilon$

$$
\begin{array}{llllll}
y=1 & \nu=-\dfrac{1}{9} & a_1=-\dfrac{1}{9} & Q_1=1 & x=\dfrac{P_1}{Q_1}=1.88889 & \text{(Newton)} \\
y'=9 & \beta=\dfrac{2}{3} & a_2=-\dfrac{2}{27} & Q_2=\dfrac{25}{27} & x+\dfrac{P_2}{Q_2}=1.88000 & \text{(Halley)} \\
y''=12 & \gamma=\dfrac{1}{6} & a_3=-\dfrac{1}{18} & Q_3=\dfrac{141}{162} & x+\dfrac{P_3}{Q_3}=1.87943 & \text{(4th order)} \\
y'''=6 & \delta=0 & a_4=-\dfrac{11}{162} & Q_4=\dfrac{1766}{2187} & x+\dfrac{P_4}{Q_4}=1.879388 & \text{(5th order)}
\end{array}
\tag{2.20}
$$

The six decimal values of $\xi=2\cos 20^\circ$ in tables is 1.879385+.

3. The values of the partial numerators a_k and remainders r_k. We next express the functions r_k and a_k as ratios of certain determinants. If ψ is the inverse of ϕ, so that $y=\phi(x)$ implies $x=\psi(y)$ for x near the values $\xi=\psi(0)$, then ξ may be expanded in a Taylor series with remainder as follows:

$$
\xi = x + (0-y)\psi'(y) + \frac{(0-y)^2}{2!}\psi''(y) + \cdots + \frac{(0-y)^k}{k!}\psi^{(k)}(\theta_k y), \tag{3.1}
$$

where the number θ_k satisfies $0<\theta_k<1$.

Then if we define c_k and c_k^* by the equations

$$
c_k = \frac{(y')^k}{k!}\psi^{(k)}(y); \qquad c_k^* = \frac{(y')^k}{k!}\psi^{(k)}(\theta_k y); \qquad (c_1 = 1), \tag{3.2}
$$

we obtain the following closed expression for ϵ in terms of ν:

$$
\epsilon = c_1\nu + c_2\nu^2 + c_3\nu^3 + \cdots + c_{k-1}\nu^{k-1} + c_k^*\nu^k. \tag{3.3}
$$

From the coefficients c_j and c^* we next form the k-rowed determinants

$$(3.4)\qquad D^*_{2k-1} = \begin{vmatrix} c_1 & c_2 & \cdots & c_k^* \\ c_2 & c_3 & \cdots & c_{k+1}^* \\ \cdots & \cdots & \cdots & \cdots \\ c_k & c_{k+1} & \cdots & c_{2k-1}^* \end{vmatrix}; \qquad D^*_{2k} = \begin{vmatrix} c_2 & c_3 & \cdots & c_{k+1}^* \\ c_3 & c_4 & \cdots & c_{k+2}^* \\ \cdots & \cdots & \cdots & \cdots \\ c_{k+1} & c_{k+2} & \cdots & c_{2k}^* \end{vmatrix}.$$

We modify D_k^* to define the three related determinants D_k, E_k^*, and E_k by replacing each c_j^* in the last column respectively by c_j, c_{j+1}^*, and c_{j+1}. Then since $c_{j+1}^* = (c_j^* - c_j)/v$ by (3.3), we obtain

$$(3.5)\qquad E_k^* = (D_k^* - D_k)/v, \qquad k \geqq 1.$$

It is convenient to make the special definitions

$$(3.4a)\qquad D_0^* = D_0 = D_{-1} = -D_{-2} = 1; \qquad E_0^* = 0.$$

Then we may prove two important identities as a lemma from which Theorem 2 will be easily obtained.

LEMMA 1. *The determinants D_k^*, D_k, E_k^*, and E_k defined in* (3.4) *and* (3.5) *satisfy the identities*:

$$(3.6)\qquad D_{k-2}D^*_{k+1} = -D_kE^*_{k-1} + D_{k-1}E^*_k, \qquad \text{for } k > 1$$

$$(3.7)\qquad D_{k-2}D_{k+1} = -D_kE_{k-1} + D_{k-1}E_k, \qquad \text{for } k > 1.$$

Proof: Let M, A, B, B^* be rectangular submatrices of (3.4) so chosen that

$$D_{k-2} = |\overline{M}| \qquad D_k = \begin{vmatrix} \overline{M} & \overline{B} \\ A & c_k \end{vmatrix} \qquad E_k^* = \begin{vmatrix} \overline{M} & \overline{B}^* \\ A & c_{k+1}^* \end{vmatrix}$$

$$D_{k+1} = \begin{vmatrix} {}'M & B & B^* \\ {}'A & c_k & c_{k+1}^* \end{vmatrix} \qquad D_{k-1} = |{}'M \quad B| \qquad E_{k-1}^* = |{}'M \quad B^*|$$

where if k is odd, $\overline{MB\overline{B}^*} = MBB^*$ *etc.*, but if k is even, $\overline{MB\overline{B}^*}$ is obtained from MBB^* by stripping off the top row; and where ${}'M = M$ and ${}'A = A$ if k is even, but ${}'M$ and ${}'A$ are obtained from M and A by stripping off the first column, if k is odd. Then by the Laplace expansion of determinants and some simple additions of rows and subtraction of columns we have

$$D_{k-2}D^*_{k+1} = \begin{vmatrix} {}'M & B & 0 & B^* \\ 0 & 0 & \overline{M} & 0 \\ {}'A & c_k & A & c_{k+1}^* \end{vmatrix} = \begin{vmatrix} {}'M & B & 0 & B^* \\ {}'\overline{M} & \overline{B} & \overline{M} & \overline{B}^* \\ {}'A & c_k & A & c_{k+1}^* \end{vmatrix}$$

$$= \begin{vmatrix} {}'M & B & 0 & B^* \\ 0 & \overline{B} & \overline{M} & \overline{B}^* \\ 0 & c_k & A & c_{k+1}^* \end{vmatrix} = \begin{vmatrix} D_{k-1} & E^*_{k-1} \\ D_k & E_k^* \end{vmatrix}.$$

This proves the identity (3.6). A simple replacement of c_j^* by c_j changes (3.6) into (3.7), and completes the proof of the lemma.

THEOREM 2. *The continued fraction partial numerators a_k and remainders r_k are expressible in terms of the determinants D_k and D_k^* of* (3.4) *by the formulas*

$$r_k = -\nu \frac{D_{k-3}D_k^*}{D_{k-2}D_{k-1}^*}, \qquad a_k = -\nu \frac{D_{k-3}D_k}{D_{k-2}D_{k-1}}. \tag{3.8}$$

Proof: When we multiply the identity (3.6) by $-\nu$ and eliminate E_{k-1}^* and E_k^* by (3.5), we obtain

$$-\nu D_{k-2}D_{k+1}^* = D_kD_{k-1}^* - D_{k-1}D_k^*. \tag{3.9}$$

Dividing (3.9) by its last term we then have

$$-\nu \frac{D_{k-2}D_{k+1}^*}{D_{k-1}D_k^*} = \frac{D_kD_{k-1}^*}{D_{k-1}D_k^*} - 1. \tag{3.10}$$

The special definitions (3.4a) were so chosen that for $k=1$ equation (3.8) becomes $r_1=\nu D_1^*=\epsilon$; $a_1=\nu D_1=\nu$. Using an induction proof, we next assume that (3.8) correctly defines $r_1, \cdots, r_k$ and $a_1, \cdots, a_k$ in accordance with the requirements of (2.1) and (2.2), and we show that (3.10) then implies the validity of (3.8) when k is replaced by $k+1$. The right member of (3.10) equals $(a_k/r_k)-1$, by (3.8). This in turn is equal to r_{k+1} by the definition (2.1). Hence the left member of (3.10) equals r_{k+1}, and the first formula of (3.8) is established for $k+1$. The formula for a_{k+1} is obtained from r_{k+1} by taking the first term in the expansion of the latter in a power series in ν. This agrees with the second formula of (3.8).

From (3.8) we next obtain simple expressions for the products R_k and A_k defined in (2.6) and (2.5). These are:

$$R_k = \frac{\nu^k D_k^*}{D_{k-2}}, \qquad A_k = \frac{\nu^k D_k}{D_{k-2}}. \tag{3.11}$$

Furthermore, from (3.8) and (3.7) we derive the formulas

$$-\frac{a_{k+1}}{a_1} = \frac{E_k}{D_k} - \frac{E_{k-1}}{D_{k-1}}, \qquad \text{(for } k \geqq 1, \text{ if we set } E_0 = 0). \tag{3.12}$$

Upon summing from 1 to k, we obtain a sum of a's (called S_k) defined by

$$S_k \equiv a_2 + a_3 + \cdots + a_{k+1} = -a_1E_k/D_k. \tag{3.13}$$

4. The derivative of the kth approximant. The error in using $f_k(x)=x+P_k/Q_k$ as an approximation to the root ξ of $\phi(x)=0$ can be estimated by means of a

surprisingly simple expression for the derivative $f_k'(x)$, which will be stated as Theorem 3. Our proof of this theorem depends on finding expressions for the derivatives with respect to x of the functions D_k, A_k, a_k, Q_k, and P_k.

LEMMA 2. *The derivatives D_k', A_k', a_k', Q_k' and P_k', (with respect to x), of the functions defined in* (3.4), (2.5), (3.8) *and* (2.3) *are given by the formulas*

$$-a_1D_k'/D_k = (k+1)S_k - [\tfrac{1}{2}(k+1)^2]a_2 \tag{4.1}$$

$$-a_1A_k'/A_k = k - (k-1)S_{k-2} + (k+1)S_k \tag{4.2}$$

$$-a_1a_k'/a_k = 1 + a_{k-1} + a_k - (k-1)a_{k-1} + (k+1)a_{k+1}, \qquad k > 1. \tag{4.3}$$

$$a_1Q_k' + S_{k-1}Q_k + (k+1)a_{k+1}a_kQ_{k-2} = 0 \tag{4.4}$$

$$a_1(P_k' + Q_k) + S_{k-1}P_k + (k+1)a_{k+1}a_kP_{k-2} = 0. \tag{4.5}$$

Proof: The elements c_{i+j-1} and c_{i+j} in the ith row and jth column of the determinants D_{2k-1} and D_{2k} were defined in (3.2). The derivative c_k' is seen to be

$$\begin{aligned} c_k' &= (y')^k y'\psi^{(k+1)}(y)/k! + k(y')^{k-1}y''\psi^{(k)}(y)/k! \\ c_k' &= (k+1)c_{k+1} - 2c_2kc_k, \qquad (\text{where } 2c_2 = -y''/y' = -2a_2/a_1). \end{aligned} \tag{4.6}$$

The derivative of a kth order determinant such as D_{2k-1} is the sum of k^2 terms obtained by multiplying the derivative d_{ij}' of each element $d_{ij}(=c_{i+j-1})$ by its cofactor C_{ij}. Each derivative c_{i+j-1}' is separated into two terms by (4.6), the first of which involves the element $c_{i+j}=d_{i,j+1}$ in the next column. Thus

$$D_{2k-1}' = \sum_{i,j=1}^{k} (i+j)c_{i+j}C_{ij} - 2c_2 \sum_{i,j=1}^{k} (i+j-1)c_{i+j-1}C_{ij}. \tag{4.7a}$$

Since $C_{ji}=C_{ij}$, an interchange of i and j will replace a term with coefficient i by an equal term with coefficient j. Recalling the definition of E_{2k-1} after (3.4), we transform (4.7a) to

$$\begin{aligned} D_{2k-1}' &= 2kE_{2k-1} + \sum_{j=1}^{k-1} 2j \sum_{i=1}^{k} d_{i,j+1}C_{ij} - 2c_2 \sum_{j=1}^{k} (2j-1) \sum_{i=1}^{k} d_{ij}C_{ij} \\ &= 2kE_{2k-1} + \sum_{j=1}^{k-1} 2j(0) \qquad - 2c_2 \sum_{j=1}^{k} (2j-1)D_{2k-1}. \end{aligned} \tag{4.7b}$$

With (3.13) in mind we multiply (4.7b) by $-a_1/D_{2k-1}$ and obtain

$$-a_1D_{2k-1}'/D_{2k-1} = 2kS_{2k-1} - 2a_2 \sum_{j=1}^{k} (2j-1). \tag{4.8a}$$

By a similar argument applied to the derivative of D_{2k} we obtain

$$-a_1D_{2k}'/D_{2k} = (2k+1)S_{2k} - 2a_2 \sum_{j=1}^{k} (2j). \tag{4.8b}$$

Use of the square brackets in (4.1) to denote the greatest integer in $\frac{1}{2}(k+1)^2$,

enables us to express the two formulas (4.8a) and (4.8b) as a single formula (4.1).

By differentiating $\ln A_k$ as obtained from (3.11) we then have

$$\begin{aligned} A_k'/A_k &= ky'/y - ky''/y' - D_{k-2}'/D_{k-2} + D_k'/D_k \\ -a_1A_k'/A_k &= k + 2ka_2 + a_1D_{k-2}'/D_{k-2} - a_1D_k'/D_k. \end{aligned} \tag{4.9}$$

Equation (4.2) then follows immediately from (4.9) and (4.1).

Next, since $a_k = -A_k/A_{k-1}$, we obtain (4.3) from (4.2) by subtraction as follows.

$$\begin{aligned} -a_1a_k'/a_k &= -a_1A_k'/A_k + a_1A_{k-1}'/A_{k-1} \qquad (k > 1), \\ &= 1 - S_{k-3} + S_{k-1} - (k-1)(S_{k-2} - S_{k-3}) + (k+1)(S_k - S_{k-1}) \\ &= 1 + a_{k-1} + a_k - (k-1)a_{k-1} + (k+1)a_{k+1}. \end{aligned} \tag{4.10}$$

We prove equation (4.4) and (4.5) by induction, using the definitions (2.3) and (3.11) .We verify directly that both left members vanish for $k=1$ and $k=2$ in view of the definitions $P_{-1}=Q_0=Q_1=1$, $Q_{-1}=P_0=S_0=0$, and the values of $a_1'=P_1'=P_2'$ and $a_2'=Q_2'$ given by (4.3).

Then taking $k=n>2$, we assume that the left members of (4.4) and (4.5) both vanish for $k=n-1$ and for $k=n-2$, and we use the recurrence relations (2.3) to express Q_n' and P_n' in terms of the known quantities Q_{n-1}', Q_{n-2}' and P_{n-1}', P_{n-2}'. From (4.4) and (4.3) we obtain successively the relations

$$\begin{aligned} 0 &= a_n\left[a_1Q_{n-2}' + S_{n-3}Q_{n-2} + (n-1)a_{n-1}a_{n-2}Q_{n-4}\right] \\ &= a_1(a_nQ_{n-2})' - a_1a_n'Q_{n-2} + S_{n-3}a_nQ_{n-2} + (n-1)a_na_{n-1}(Q_{n-2} - Q_{n-3}) \\ &= a_1(Q_n' - Q_{n-1}') - (n-1)a_na_{n-1}Q_{n-3} \\ &\qquad + a_nQ_{n-2}(-a_1a_n'/a_n + S_{n-3} + (n-1)a_{n-1}) \\ &= a_1Q_n' - a_1Q_{n-1}' - (n-1)a_na_{n-1}Q_{n-3} + a_nQ_{n-2}(1 + S_{n-1} + (n+1)a_{n+1}) \\ &= a_1Q_n' + S_{n-2}Q_{n-1} + a_na_{n-1}Q_{n-3} + a_nQ_{n-2} + a_nQ_{n-2}(S_{n-1} + (n+1)a_{n+1}) \\ &= a_1Q_n' + (S_{n-2} + a_n)Q_{n-1} + S_{n-1}a_nQ_{n-2} + (n+1)a_{n+1}a_nQ_{n-2} \\ &= a_1Q_n' + S_{n-1}Q_n + (n+1)a_{n+1}a_nQ_{n-2}. \end{aligned} \tag{4.11}$$

Hence equation (4.4) is established for $k=n$ and the induction proof is complete. By a series of steps similar to (4.11) we may prove (4.5) by induction and thus finish Lemma 2.

To find the derivative of P_k/Q_k we first multiply (4.5) by Q_k and (4.4) by P_k and subtract. This gives us the relation

$$a_1(Q_k^2 + Q_kP_k' - P_kQ_k') - (k+1)a_{k+1}a_k(P_kQ_{k-2} - Q_kP_{k-2}) = 0. \tag{4.12}$$

When we make use of the values of A_{k-1} and A_{k+1} in (2.5), equation (4.12) becomes

$$Q_k^2 + Q_k P_k' - P_k Q_k' = (k+1)A_{k+1}/a_1. \tag{4.13}$$

Division by Q_k^2 in (4.13) gives the desired derivative of $x+P_k/Q_k$.

THEOREM 3. *The derivative of the kth iteration function $f_k(x)=x+P_k/Q_k$, defined by the kth approximant P_k/Q_k to the continued fraction* (1.3) *is given by the following formula of Tauber* [14]:

$$f_k'(x) = \left(x + \frac{P_k}{Q_k}\right)' = \frac{(k+1)A_{k+1}}{Q_k^2 a_1} = \frac{k+1}{Q_k^2}\,\frac{D_{k+1}}{D_{k-1}}\,\nu^k. \tag{4.14}$$

Furthermore, the ratio ϵ_k/ϵ of the residual error $\xi - f_k(x) = \epsilon_k$ to the initial error $\xi - x = \epsilon$ is less in absolute value than the maximum value of $f_k'(x)$ on an interval U containing x and ξ. Also

$$\left|\frac{\epsilon_k}{\epsilon^{k+1}}\right| < \max_{x\in U} Q_k^{-2} \prod_{s=2}^{k+1}\left|\frac{a_s}{\xi - x}\right| = \max_{x\in U} Q_k^{-2}\left(\frac{\nu}{\epsilon}\right)^k \left|\frac{D_{k+1}}{D_{k-1}}\right|. \tag{4.15}$$

Proof: Formula (4.14) follows immediately from (4.13) and (3.11). When we integrate (4.14) from x to ξ we obtain

$$\begin{aligned}\frac{\epsilon_k}{\epsilon} &= \frac{f_k(\xi) - f_k(x)}{\epsilon} \\ &= \frac{1}{\epsilon}\int_{\xi-\epsilon}^{\xi} [f_k'(t)]\,dt = \frac{1}{\epsilon}\int_{t-\xi-\epsilon}^{t-\xi}\left[Q_k^{-2}\prod_{s=2}^{k+1}\left(\frac{-a_s}{t-\xi}\right)\right] d(t-\xi)^{k+1}.\end{aligned} \tag{4.16}$$

By extracting from this integral the maximum value of the quantity in brackets in the interval of integration, we obtain the upper bounds for $\left|\epsilon_k/\epsilon^{k+1}\right|$ given in Theorem 3.

A good estimate of the residual error ϵ_k is $(D_{k+1}/D_{k-1})\nu^{k+1}$, since Q_k and ν/ϵ are near 1 when ϵ is small. The estimate is especially good for $k \geqq 2$ if the trial value x is chosen on that side of ξ for which y and y'' have opposite signs, so that $\beta > 0$ and $Q_2 > 1$.

5. The case of positive numerators a_k. There is a class of equations for which the kth approximant P_k/Q_k to the continued fraction (1.3) always gives a closer estimate for ϵ than the kth partial sum s_k of the corresponding power series. In this class fall the familiar quadratic equations

$$\phi(\xi) = 0, \text{ where } \phi(t) \equiv t^2 - 2At + B = (t-x)^2 + (1/\beta)(t-x) - \nu/\beta. \tag{5.1}$$

The constants ν and β may assume complex values but have the same meaning as in (1.2), and the real or complex constant x is the chosen first estimate for the root ξ. Writing $\xi - x = \epsilon$ and $\nu\beta = a$, our quadratic equation becomes

$$(\beta\epsilon)^2 + (\beta\epsilon) - a = 0. \tag{5.2}$$

The power series solution of the form (3.3) is

$$\beta\epsilon = \tfrac{1}{2}(-1+\sqrt{1+4a}) = a - a^2 + \frac{2\cdot 6}{2\cdot 3}a^3 - \frac{2\cdot 6\cdot 10}{2\cdot 3\cdot 4}a^4 + \cdots \tag{5.3}$$

and the corresponding continued fraction is

$$\beta\epsilon = \cfrac{a}{1+\cfrac{a}{1+\cfrac{a}{1+\cdots}}} \qquad \text{where } a_k = a \text{ and } r_k = \beta\epsilon, \text{ for } k > 1. \tag{5.4}$$

The approximants P_k/Q_k for (5.4) are given explicitly by the formulas

$$Q_{k-1} = (\beta/a)P_k = [(1+\beta\epsilon)^k - (-\beta\epsilon)^k]/(1+2\beta\epsilon) \tag{5.5}$$

so the residual error of the kth approximant is

$$\epsilon_k = \epsilon - P_k/Q_k = \frac{(1/\beta)+2\epsilon}{1-\left(\dfrac{1+\beta\epsilon}{-\beta\epsilon}\right)^{k+1}}. \tag{5.6}$$

Thus for all complex values of x except those equally distant from the two roots of $\phi(\xi)=0$, the continued fraction (5.4) converges to the nearer root ξ [17], since $|-\epsilon|$ and $|1/\beta+\epsilon|$ are the respective distances from x to the nearer and further root of (5.1). However, if the coefficients A and B in (5.1) are fixed but x varies over the complex plane, the series (5.3) converges only if x lies inside the rectangular hyperbola with foci at the roots.

Not only does the continued function converge in a larger region than the power series, but the convergence is actually better as we see from the following theorem.

THEOREM 4. *Let the real function $\phi(x)$ and a real root ξ of $\phi(\xi)=0$ be such that the k ratios of partial numerators $b_j=a_j/a_1$ $(j=2, 3, \cdots, k+1)$ all have the same sign in a neighborhood U of ξ, and let x be chosen on that side of ξ for which $a_2, a_3, \cdots, a_{k+1}$ are all positive. Then for $k>1$ the kth approximant P_k/Q_k to the continued fraction* (1.3) *gives a closer approximation to the error $\epsilon=\xi-x$ than does the sum s_k of the first k terms $c_j\nu^j$ of the series* (3.3). *Furthermore the first $k+1$ terms of the series will then alternate in sign.*

Proof: The signs of ϵ and ν can be changed without altering $a_2, a_3, \cdots, a_{k+1}$ by replacing x by $2\xi-x$ and thus reversing the direction of the x axis. Hence it will be sufficient to prove the theorem in the case that ν and b_j are negative and a_j and r_j are positive between x and ξ, for $j=2, 3, \cdots, k+1$. From equation (2.1) or (1.3) we obtain expansions for the r_k in powers of ξ.

$$(5.7) \qquad r_{m-1} = \frac{a_{m-1}}{1 + r_m} = a_{m-1} \sum_{t=0}^{\infty} (-r_m)^t = \nu b_{m-1} \sum_{t=0}^{\infty} \nu^t (-r_m/\nu)^t, \qquad m > 2.$$

The remainder r_k vanishes when $\nu = 0$ and has two negative coefficients b_k and $-b_k b_{k+1}$ in its expansion in powers of ν obtained from (5.7). Likewise r_{k-1} has three negative coefficients in its expansion, and keeps these even when r_k is replaced by a_k and ϵ by P_k/Q_k. From (5.7) it follows that the assumption that r_m has $k+2-m$ negative coefficients implies that r_{m-1} has at least one more, $(m>2)$. Hence by induction r_{k+1-j} has at least $j+1$ negative coefficients $(j=1, 2, \cdots, k-1)$. Thus $r_1 = \epsilon$ has at least $k+1$ positive coefficients, since $b_1 > 0$. Since ν is negative, the values of the first $k+1$ terms in the power series expansion of both ϵ and P_k/Q_k must alternate in sign. The correction R_{k+1}/Q_k of (2.7) contains ν^{k+1} as a factor and has the same sign as $(-1)^k \nu$ and $c_{k+1}\nu^{k+1}$. Hence the series expansions of ϵ and P_k/Q_k agree to k terms and we have

$$(5.8) \qquad |\epsilon - s_k| = |R_{k+1}/Q_k| + |P_k/Q_k - s_k| > |R_{k+1}/Q_k| = |\epsilon_k|, \quad \text{for } k > 1.$$

Thus whenever the partial numerators $a_2, \cdots, a_{k+1}$ are all positive, the continued fraction error ϵ_k is numerically less than the series remainder $\epsilon - s_k$.

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THE SEMI-ORTHOCENTRIC TETRAHEDRON

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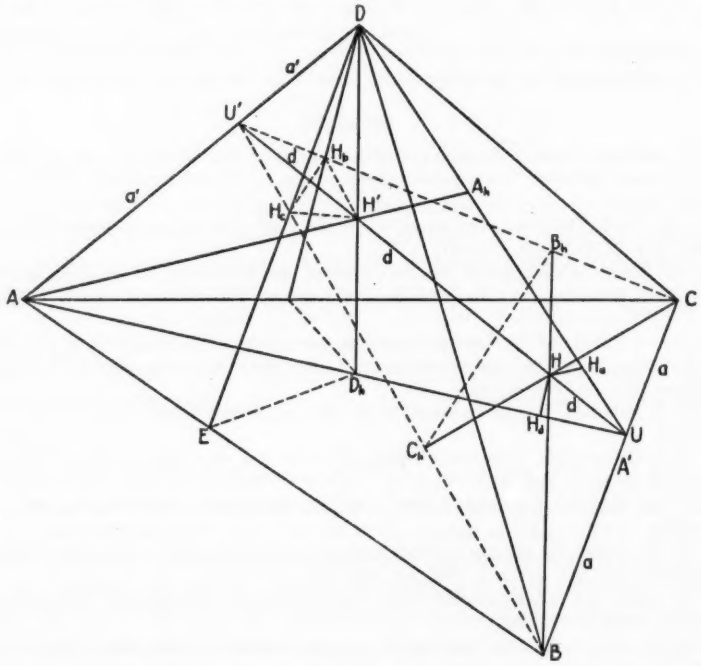
The altitudes of a tetrahedron form, in general, a hyperbolic group of four lines.

The four altitudes have a point in common, if, and only if, each edge of the tetrahedron is perpendicular to the opposite edge. The tetrahedron is then said to be orthocentric, and the common point of the altitudes is its orthocenter.

The intermediate case of a tetrahedron with one pair of opposite orthogonal edges has so far received scant attention. Such a tetrahedron may be said to be *semi-orthocentric*. It shall be denoted by K in what follows.

The two rectangular edges of K may be referred to as its *principal* or *special* edges, and their common perpendicular as the *principal bialtitude* of K. The four remaining edges of K will be said to be *non-special*.



1. The Preliminaries. a. Let $a = BC$, $a' = DA$ be the principal edges of a semi-orthocentric tetrahedron $K = DABC$, and U, U' the feet, on a, a', respectively, of the principal bialtitude $d = UU'$.

The edges a, a' are, respectively, perpendicular to the planes $a'd = DAU$, $ad = BCU'$, and the two planes ad, $a'd$ are mutually perpendicular.

b. It is known that the two pairs of altitudes BB_h, CC_h, and DD_h, AA_h of K lie in the planes ad, $a'd$, respectively, and intersect on the principal bialtitude d of K. Conversely, if two altitudes of a tetrahedron intersect, the same holds for the other two altitudes, and the tetrahedron is semi-orthocentric [1; arts. 206, 204, p. 62].

c. The two points $H=(BB_h, CC_h)$, $H'=(DD_h, AA_h)$ will be referred to as the *semi-orthocenters* of K, relative to the principal edges a, a', respectively.

2. The mid-points of the non-special edges of K. a. The non-special edges of K form a skew quadrilateral $DBAC$ whose diagonals are the special edges DA, BC. Thus the parallelogram formed by the mid-points of the sides of this skew quadrilateral [1; p. 42] is a rectangle. Its diagonals are the two bimedians of K relative to the two pairs of non-special opposite edges of K, hence those two bimedians are equal, and the center of the rectangle is the centroid of K.

Moreover, the mediator (that is, the perpendicular bisecting plane) ρ of the principal bialtitude UU' being parallel to and equidistant from the two principal edges DA, BC of K, the vertices of the rectangle considered lie in the plane ρ [1; p. 3, art. 10]. To sum up: *In a semi-orthocentric tetrahedron K the mid-points of four non-special edges of K lie in the mediator of the principal bialtitude and are the vertices of a rectangle whose circumcenter is the centroid of K.*

b. Conversely. *If the mid-points of two pairs of opposite edges of a tetrahedron are the vertices of a rectangle, the tetrahedron is semi-orthocentric.*

3. The nine-point circles of the faces of K. a. The line $a=BC$ of K being perpendicular to the plane ADU of U (§1a), the lines DU, AU are the altitudes of the triangles DBC, ABC, to their common base BC. Conversely, if in a tetrahedron $DABC$ the altitudes of the triangles DBC, ABC, to their common base BC, meet that base in the same point U, the line BC is perpendicular to the plane DAU, and therefore to the line AD, that is, the tetrahedron is semi-orthocentric.

Thus in the tetrahedron $K=DABC$ the nine-point circles of the faces DBC, ABC have two points in common, namely the point U and the mid-point A' of the principal edge BC, hence: *The nine-point circles of two faces of a tetrahedron are cospherical, if, and only if, the tetrahedron is semi-orthocentric, and the common edge of the two faces is a principal element of the tetrahedron.*

b. As an immediate consequence we have: *If the nine-point circles of two faces of a tetrahedron are cospherical, the same is true of the nine-point circles of the remaining two faces of the tetrahedron.*

c. We thus associate with a semi-orthocentric tetrahedron K two spheres (G_a), (G_a') relative to the two principal elements a, a' of K.

These two spheres will be referred to as the *G-spheres* of K.

d. The sphere (G_a) containing the nine-point circles of the faces DBC, ABC, passes through the mid-points of the non-special edges DB, DC and AB, AC. The sphere (G_a') passes through the same four points, for analogous reasons.

Hence (§2a): *The radical plane of the two G-spheres of a semi-orthocentric tetrahedron K is the mediator of the principal bialtitude of K.*

The line of centers of the two G-spheres is thus parallel to the principal bialtitude of K, and, furthermore, passes through the centroid of K (§2a).

e. Given a tetrahedron $DABC$, if the perpendiculars to the faces DBC, ABC at their nine-point centers meet in a point G_a, the sphere having G_a for center and passing through the midpoint A' of the edge BC will pass through the nine-point circles of those two faces, hence (§3a): *The perpendiculars erected to two faces of a tetrahedron as their nine-point centers are coplanar, if, and only if, the tetrahedron is semi-orthocentric, and the common edge of the two faces considered is a principal edge.*

4. The orthocenters of the faces of K. a. If from the foot D_h of the altitude DD_h of K a perpendicular D_hE is drawn to the line AB, the line DE is an altitude of the triangle DAB.

The plane DAB intersects the plane $ad=BCU'$ along the line BU'. Now the special edge $a'=DA$ is perpendicular to the plane ad (§1a), hence DA is perpendicular to $U'B$. Thus the point $H_c=(BU', DE)$ is the orthocenter of the triangle DAB.

The point H_c belongs to each of the planes DD_hE, $ad=BCU'$, and so does the semi-orthocenter H' of K (§§1b, 1c), hence $H'H_c$ is the line of intersection of those two planes. Now the plane DAB passes through the lines BA, DA which are, respectively, perpendicular to the planes DD_cE, $ad=BCU'$, hence the line $H'H_c$ is the perpendicular to the face DAB at its orthocenter H_c. Similarly for the orthocenters H_a, H_b, H_d of the faces BCD, CDA, ABC. of K. Thus: *In a semi-orthocentric tetrahedron K the semi-orthocenter relative to a given principal edge of K is the point of intersection of the perpendiculars erected to the two faces of K passing through that special edge, at the orthocenters of those faces.*

b. Conversely. *If the perpendiculars to two faces of a tetrahedron at their respective orthocenters intersect, the tetrahedron is semi-orthocentric.*

If in a tetrahedron $DABC$ the perpendiculars to the faces BCA, BCD at their orthocenters H_d, H_a intersect, in a point H, the plane HH_aH_d is perpendicular to each of the planes BCA, BCD, and therefore to their line of intersection BC, at a point U. The line UH_d is thus the altitude of the triangle ABC relative to the side BC and therefore passes through its vertex A. The line UH_a passes through the vertex D, for analogous reasons.

Thus the edge DA of the tetrahedron lies in the plane HH_aH_d perpendicular to BC, hence the two edges DA, BC are perpendicular, which proves the proposition (1a).

5. The polar circles of the faces of K. a. Let (H) be the sphere having the point H for center and orthogonal to the sphere (BC) having the principal edge BC for diameter.

The plane ABC cuts the sphere (BC) along a great circle (bc) having BC for diameter, and the sphere (H) along a small circle η orthogonal to (bc) [1; p. 176,

ex. 19] whose center is the foot H_d of the perpendicular HH_d from H to the plane ABC. Now the polar circle (h_d) of the triangle ABC has for center the orthocenter H_d, and (h_d) is orthogonal to the circle (bc) having the side BC of the triangle ABC for diameter, hence (h_d) coincides with the circle η, that is, (h_d) lies on the sphere (H). The same holds for the polar circle (h_a) of the triangle DBC, for analogous reasons.

Similar considerations may be applied to the sphere (H') having H' for center and orthogonal to the sphere (DA) having DA for diameter. Thus: *In a semi-orthocentric tetrahedron K the polar circles of the two faces of K passing through the same principal edge lie on a sphere having for center the semi-orthocenter of K relative to the edge considered.*

b. *Definition.* We thus associate with a semi-orthocentric tetrahedron K two spheres (H), (H') which may be referred to as the *semi-polar spheres* of (K). They may be said to be relative to the principal elements $a=BC$, $a'=DA$, respectively.

c. Conversely. *If the polar circles of two faces of a tetrahedron are cospherical, the tetrahedron is semi-orthocentric.*

Indeed, if two polar circles are cospherical, their axes meet in the center of that sphere, that is, the perpendiculars to the two faces of the tetrahedron at their orthocenters are coplanar, hence the proposition (§4b).

6. The semi-polar spheres of K. a. The ends of the diameter BC of the sphere (BC) orthogonal to the sphere (H) (§5) are conjugate points with respect to (H) [1; p. 164, art. 413]. Thus the polar plane of the point B with respect to the sphere (H) passes through the point C and is perpendicular to the line $BH=BB_h$ joining B to the center H of (H), hence that plane coincides with the face CDA of K. Similarly the vertex C and the face BDA are pole and polar plane for the sphere (H).

Analogous considerations apply to the vertices D, A and the sphere (H') of K. Thus: *The two vertices of a semi-orthocentric tetrahedron K lying on a special edge of K have for their polar planes, with respect to the semi-polar sphere of K relative to the edge considered, the two respectively opposite faces of K.*

b. The two lines $a=BC$, $a'=DA$ are thus polar reciprocal with respect to the sphere (H) and meet their common perpendicular $d=UU'$ in two points U, U' inverse with respect to (H) [1; p. 144, art. 443]. The points U, U' are also inverse with respect to (H'), for analogous reasons. Consequently: i. *The two semi-polar spheres of a semi-orthocentric tetrahedron K have for limiting points the ends of the principal bialtitude of K* [1; p. 179]. ii. *The radical plane of the two semi-polar spheres of K is the mediator of the principal bialtitude of K* (§2).

7. The Monge point of K. The perpendicular to the face ABC of K at the mid-point of the segment D_hH_d (§4) passes through the Monge point of K [1; p. 70, art. 233]. Similarly for the analogous perpendiculars to the other faces of K. Now each of those four perpendiculars passes through the mid-point M of the segment HH'; hence: *In a semi-orthocentric tetrahedron K the Monge point*

lies on the principal bialtitude and bisects the segment determined by the semi-orthocenters of K.

8. The orthocentric tetrahedron. An orthocentric tetrahedron may be considered as a semi-orthocentric tetrahedron which has three pairs of "principal" edges (§1). The basic properties of an orthocentric tetrahedron may be derived from the properties of a semi-orthocentric tetrahedron established in the preceding paragraphs. Here is an example.

In an orthocentric tetrahedron (L) the three edges of any face are "principal" edges, hence the perpendicular to any face of (L) at the nine-point center of that face meets the analogous perpendiculars relative to the remaining three faces of (L). Now these four perpendiculars cannot lie in one plane (for that would imply that the four faces of (L) are perpendicular to that plane), hence they have a point in common.

Thus the three pairs of G-spheres (§3) relative to the three pairs of opposite edges of (L) have a common center. This center is therefore the centroid of (L) and the six spheres coincide in one sphere which passes through the nine-point circles of the four faces of (L) [1; pp. 261, 262].

Reference

1. Nathan Altshiller-Court, Modern Pure Solid Geometry. New York, 1935.

OPTIMAL HORSE RACE BETS

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1. Introduction. The problem of placing straight win bets on a horse race so as to maximize the expected value of the profit is intellectually lucrative, although fiscally academic unless the bettor is extraordinarily opulent, has an excellent dopester, and can do elaborate computations on very short notice.

The crowd bets on the various horses. The total sum, after a fixed percentage being deducted as the track's profit, is allocated to the bettors on the winning horse in proportion to their bets. We assume that we know the true probability of each horse's winning.

There is nothing unsound or new in our basic wagering principle. It is known, in track parlance, as "overlays." It consists in taking advantage of the collective error of the crowd in appraising the probabilities as registered by the amounts they bet. What we believe is new is the exact solution.

Let us take an example. The third column is the amounts bet by the crowd, drastically scaled down from race track reality.

Horse	Probability of Winning	Amount Bet by Crowd
A	.4	\$1000
B	.2	350
C	.2	300
D	.1	250
E	.1	100
	1.0	\$2000

If we neglect, for the moment, the track's deduction, it is clear that E is the soundest investment; for he returns twenty times the stake and has a win probability of 0.1. But if our bet on him is large, the increment to the \$100 will affect the entire picture and possibly even cause another horse to be the most favorable. We now can see that to maximize our expected profit we should place bets on a certain subset (possibly null) of horses.

Since this bagatelle first resulted from a conversation with a turf-minded friend, we have learned of its wider interpretations. It is akin to the current field of non-linear programming whose advocates have suggested alternative techniques. But in such fields, solutions by such definite means as algorithm and formula, as here obtained, are uncommon and we have been told our work should hold some exoteric interest. We acknowledge, with admiration, the suggestions of the referee who elegantly abridged our derivation.

2. The problem. Let the win probability of the jth horse be p_j, the amount wagered on him by others than ourself s_j, and our wager on him be x_j. Let the total sum wagered be multiplied by Q $(0<Q<1)$ prior to distribution to the winning bettors. Our problem then is:

What values of

$$x_j \,(x_j \geqq 0,\ j = 1, \cdots, n)$$

render the expected value of our profit,

$$(1) \qquad F(x_1, \cdots, x_n) = Q\left[\sum_{j=1}^{n} (x_j + s_j)\right]\sum_{j=1}^{n} \frac{p_j x_j}{x_j + s_j} - \sum_{j=1}^{n} x_j$$

a maximum, the value of the maximal F to be positive? Here Q and p_j, s_j are constants subject to

$$0 \leqq p_j \leqq 1, \qquad \sum_{j=1}^{n} p_j = 1,$$

$$s_j > 0,^* \qquad 0 < Q < 1,$$

and, of course, $n > 1$.

3. The solution. Let c_j $(j=1, \cdots, n)$ be such that

$$c_j \geqq 0, \qquad \sum_{j=1}^{n} c_j = 1.$$

If we put $x_j = c_j u$ in (1), F becomes a function of u denoted by $f(u)$. It is easy to verify.

$$f(0) = 0 \tag{2}$$

$$\lim_{u \to \infty} \frac{f(u)}{u} \leqq Q - 1 < 0 \tag{3}$$

$$f'(0) = \sum_{j=1}^{n} c_j \left[Q \frac{p_j}{s_j} \sum_{i=1}^{n} s_i - 1 \right]. \tag{4}$$

Let u range from 0 to ∞. From (2) and (3) we see that f begins as 0 but ultimately becomes and remains negative. If at least one of the brackets of (4) is positive, the c_j can be chosen so that $f'(0) > 0$. Thus we state:

LEMMA 1. *There is always a non-negative maximum of F; if*

$$\max_{1 \leqq j \leqq n} \frac{p_j}{s_j} > \frac{1}{Q \sum_i s_i} \tag{5}$$

then this maximum is positive.

Let $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n)$ furnish a maximum. If $\bar{x}_i > 0$, then

$$\left(\frac{\partial F}{\partial x_i} \right)_{X = \overline{X}} = Q \sum_{j=1}^{n} \frac{p_j \bar{x}_j}{\bar{x}_j + s_j} + \frac{p_i s_i}{(\bar{x}_i + s_i)^2} Q \sum_{j=1}^{n} (\bar{x}_j + s_j) - 1 = 0. \tag{6}$$

Then $p_i s_i / (\bar{x}_i + s_i)^2$ has the same value for all i such that $\bar{x}_i > 0$ and we call it $1/\lambda^2$. Thus for such i

$$\bar{x}_i = \lambda \sqrt{p_i s_i} - s_i. \tag{7}$$

LEMMA 2. *The maximal F never occurs when all the $\bar{x}_i$ are positive*; or: *Don't bet on all the horses.*

Substituting from (7) for *every* i into (6), the central member of the latter becomes $Q - 1$ which is negative.

* We will show later that there is not necessarily a solution if some s_j are zero.

Recognizing that the higher p_j is, and the lower s_j is, the better bet Horse j is, it is clear that the ratio $\rho_j = p_j/s_j$ provides a heuristic measure of the soundness of a bet on Horse j. Let us renumber the horses, if necessary, in order of increasing value by this criterion, *i.e.*, so that $\rho_1 \leqq \rho_2 \leqq \cdots \leqq \rho_n$. The next proposition tells us that, in maximizing our expectation, if we bet on a certain horse, then we bet on all horses which are as good or better bets in this ordering. It also asserts that the optimal amounts of wager are uniquely determined by the set of horses on which we bet.

LEMMA 3. *The only positive maxima of F are achieved by $\overline{X} = (\bar{x}_1, \cdots, \bar{x}_n)$ of the form*:

$$\bar{x}_1 = \cdots = \bar{x}_{t-1} = 0, \qquad \bar{x}_t > 0, \cdots, \bar{x}_n > 0 \tag{8}$$

where $\overline{X}$ is uniquely determined by t as follows:

$$\bar{x}_i = \lambda_t \sqrt{p_i s_i} - s_i \qquad \text{for } i = t, \cdots, n. \tag{7}$$

Also

$$\rho_i \leqq \frac{1}{\lambda_t^2} \text{ for } i = 1, \cdots, t-1 \tag{9}$$

where

$$\lambda_t^2 = Q \sum_{j=1}^{t-1} s_j \Big/ \left(1 - Q \sum_{j=t}^{n} p_j\right). \tag{10}$$

Remark: Lemma 2 assures us that $t > 1$, so that (10) is guaranteed a meaning.
Proof: From (6)

$$\frac{1}{\lambda^2} = \frac{1 - Q \sum_{j=1}^{n} \dfrac{p_j \bar{x}_j}{\bar{x}_j + s_j}}{Q \sum_{j=1}^{n} (\bar{x}_j + s_j)}. \tag{11}$$

Remark that $\bar{x}_i > 0$ implies

$$\rho_i > \frac{p_i s_i}{(\bar{x}_i + s_i)^2} = \frac{1}{\lambda^2} \tag{12}$$

while $\bar{x}_i = 0$ implies

$$\left(\frac{\partial F}{\partial x_i}\right)_{X = \overline{X}} \leqq 0$$

and (6) becomes

$$Q\sum_{j=1}^{n}\frac{p_j\bar{x}_j}{\bar{x}_j+s_j}+\rho_iQ\sum_{j=1}^{n}(\bar{x}_j+s_j)-1\leqq 0.$$

A glance at (11) shows that now

$$\rho_i\leqq\frac{1}{\lambda^2}. \tag{9}$$

Hence (8) is the only possible form for a solution.

Suppose (8) to hold for some t, we calculate λ by substituting from (7) for $i=1,\cdots,t-1$ into (6). The resulting λ depends on t; it is denoted by λ_t and given by (10).

LEMMA 4. *If*

$$\rho_t>\frac{1}{\lambda_t^2},$$

then

$$\rho_t>\frac{1}{\lambda_{t+1}^2} \qquad \textit{for } t=2,\cdots,n-1.$$

For

$$\frac{1}{\lambda_{t+1}^2}=\left[\left(1-Q\sum_{j=t}^{n}p_j\right)+Qp_t\right]\Big/Q\sum_{j=1}^{t}s_j$$

and

$$1-Q\sum_{j=t}^{n}p_j=Q\sum_{j=1}^{t-1}s_j\Big/\lambda_t^2<Q\rho_t\sum_{j=1}^{t}s_j,$$

so that

$$\frac{1}{\lambda_{t+1}^2}<\left[\rho_t\sum_{j=1}^{t-1}s_j+\rho_ts_t\right]\Big/\sum_{j=1}^{t}s_j=\rho_t.$$

Let us write the inequalities

$$(K_v)\qquad \rho_v>1/\lambda_v^2=\left[q+\sum_{j=1}^{v-1}p_j\right]\Big/\sum_{j=1}^{v-1}s_j$$

for $v=2,\cdots,n$. Here $q=(1-Q)/Q$, the new form of λ_v being convenient for computation.

LEMMA 5. *If any of the* (K_v) *hold, F has a positive maximum.*

Proof: If (K_v) holds with $v<n$, then, from Lemma 4, $\rho_{v+1}\geqq\rho_v>1/\lambda_{v+1}^2$ so that (K_{v+1}) holds. Then (K_n) holds and it is

$$\rho_n>[1-Qp_n]/Q\sum_{j=1}^{n-1}s_j=\left[\frac{1}{Q}-\rho_ns_n\right]\Big/\left[\sum_{j=1}^{n}s_j-s_n\right]$$

or

$$1/Q \sum_{j=1}^{n} s_j < \rho_n = \max \frac{p_j}{s_j},$$

and we apply Lemma 1.

THEOREM. *If all of* (K_v) *are false, the* $\hat{x}_j$ *are all zero. Otherwise the* t *of Lemma* 3 *is the smallest* v *such that* (K_v) *is true.*

Proof: Suppose all the (K_v) false. If the $\hat{x}_j$ were not all zero, we would have the situation described in Lemma 3 with a $t>1$. Then (7) for $i=t$ would imply the truth of (K_t).

If any (K_v) hold, Lemma 5 assures us that t exists. Again (7) tells us that (K_t) is true. Let $t>2$ and suppose (K_{t-1}) true. Then $\rho_{t-1}>1/\lambda_{t-1}^2$ and, from Lemma 4, $\rho_{t-1}>1/\lambda_t^2$, contradicting (9) with $i=t-1$.

The calculatory procedure is now clear: We successively test the inequalities (K_v). The first true one (if any) furnishes us t and λ_t. The $\hat{x}_j$ are computed from (7).

A Remark: We show now that if some of the s_j are zero a solution need not exist. Let us take $n=2$, $s_1>0$, $s_2=0$, and $p_2>0$. Then a brief calculation shows that

$$\sup F(x_1, x_2) = F(0, 0+) = Qp_2s_1 > 0$$

while

$$F(0, 0) = 0.$$

An example: Applying our method to the above example and taking Q as 0.9, we find we should wager

\$28.40 on B
50.33 on C
43.02 on E.

4. Some ramifications. R. M. Thrall* has developed some interesting variants. The present mathematical scheme fits a problem of merchandising economics; here the restriction $Q<1$ may be dropped. Then the bigger the x_j, the bigger F and a maximum no longer exists unless the sum of the x_j be bounded.

Reverting to the original setting, he observes that after a bettor has made an optimal wager, it may still be possible for a second bettor to do likewise with a positive expected profit. It will be small and the first bettor's profit will be severely diminished. Thrall's problem is: What is our optimal bet if we know that a succeeding bettor is to wager optimally?

* R. M. Thrall, Some results in non-linear programming. To be published.

MATHEMATICAL NOTES

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THE MATRICES *AB* AND *BA*

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There has been considerable interest recently in the pair of matrices AB and BA. It is well known that, for square matrices A and B, the matrices AB and BA have the same characteristic equation [1]. The proof given here may be of interest because it is short and elementary.

If A is non-singular, we have

$$A^{I}(AB)A = BA$$

and hence the matrices have the same characteristic equation since they are similar. If both A and B are singular Roth [2] has pointed out that AB and BA need not be similar. Flanders [3] has recently shown that AB and BA have the same elementary divisors except for those corresponding to the characteristic root zero even when A and B are not square matrices.

Now write

$$F(x, y) = |Ix - (A - Iy)B| = x^n + f_1(y)x^{n-1} + \cdots + f_n(y) \tag{1}$$

and

$$G(x, y) = |Ix - B(A - Iy)| = x^n + g_1(y)x^{n-1} + \cdots + g_n(y). \tag{2}$$

Since $F(x, y)$ and $G(x, y)$ are polynomials of degree n in x and y, $f_i(y)$ and $g_i(y)$ are polynomials of degrees not greater than n. Also from the statement above $f_i(y) = g_i(y)$ whenever $|A - Iy| \neq 0$. Since these polynomials are equal for an infinite number of values of y it follows that $f_i(y) \equiv g_i(y)$ and $F(x, y) \equiv G(x, y)$. Setting $y = 0$ we have

$$|Ix - AB| \equiv |Ix - BA|.$$

The argument here is based on a field with an infinite number of elements. The referee has suggested that this is not necessary for the proof to be valid. It would suffice to have a field with more than $2n$ elements.

References

1. H. S. Thurston, On the characteristic equation of products of square matrices, this MONTHLY, vol. 38, 1931, pp. 322–324.

2. W. E. Roth, A theorem on matrices, this MONTHLY, vol. 44, 1937, p. 95.

3. Harley Flanders, Elementary divisors of AB and BA, Proceedings of the Amer. Math. Soc. vol. 2, 1951, pp. 871–874.

TWO EXAMPLES IN REAL VARIABLES

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Every Riemann integrable function is Lebesgue integrable (we speak only of bounded functions on bounded sets). Furthermore, a Riemann integrable function can be made non-integrable by redefining its values at a set of measure zero; for example, the characteristic function of the rationals is obtained in this way from the identically zero function. Our first example shows that not every Lebesgue integrable function can be obtained in this way; even though such a function can be approximated uniformly, except on a set of arbitrarily small measure, by a continuous (*a fortiori* Riemann integrable) function.

The set of Lebesgue measurable functions includes, as another subclass, the functions of Baire class I. Our second example shows that this class does not include the class of Riemann integrable functions. That it is not included in this latter class follows from example 1 and the fact that every measurable function can be obtained by redefining, on a set of measure 0, the values of a function of Baire class I. In fact, the function constructed for example 1 is itself of Baire class I.

Our examples are as follows:

1. *A bounded Lebesgue integrable function f such that there is no Riemann integrable function g with $f=g$ almost everywhere.*

2. *A Riemann integrable function which is not of Baire class* I.

1. Let E be an open set which includes all the rational numbers in the unit interval, such that $m(E)<\frac{1}{2}$, where $m(E)$ is the Lebesgue measure of E. Let f be the characteristic function of E. Clearly f is lower semi-continuous, thus of Baire class I, thus Lebesgue integrable. Suppose g is a function with $g=f$ almost everywhere. Then there is a set H of measure 0 with $g(x)=1$ for x in $E-H$, $g(x)=0$ for x in $C(E)-H$, where $C(E)$ is the complement of E.

Since E includes a dense set, E is itself dense, as is $E-H$. Thus $g(x)=1$ on a dense set, and so g is discontinuous at each point x where $g(x)\neq 1$, in particular for x in $C(E)-H$. But the measure of $C(E)-H$ is greater than $\frac{1}{2}$ and so g is not Riemann integrable.

2. Let K be the Cantor nowhere-dense perfect set. Let $\{x_n\}$ be a sequence of elements of K which is dense in K, for example the set of endpoints of the "removed intervals." $f(x)=0$ if x is not in K, $f(x)=1$ if $x=x_n$ for some n, $f(x)=2$ if x is in K but $x\neq x_n$ for all n.

Then, f is continuous on a set of measure one and so is Riemann integrable. However, there is a perfect set K such that f, considered as a function on K alone, has no point of continuity. Thus f is not of Baire class I.

Example 1 is implicit in a paper of Oxtoby, Bulletin of the American Mathematical Society, vol. 43, 1937, Th. 3, p. 247. Examples similar to Example 2 occur in the early issues of *Fundamenta Mathematicae*.

A PROOF OF A SUFFICIENT CONDITION THAT TWO SURFACES BE APPLICABLE

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On page 176 of his book [1] W. C. Graustein stated that two surfaces are applicable if and only if they can be mapped geodesically so that the total curvatures in corresponding points are equal. That the condition is necessary is obvious. The proof of the sufficiency of the condition was divided into two cases. Graustein proved one case and omitted the other, saying that it can be shown though not without difficulty. This note offers a simple complete proof with the notation of Eisenhart [2].

THEOREM. *If two surfaces can be mapped geodesically so that the total curvatures in corresponding points are equal, then the two surfaces are applicable.*

Proof. Let S and $\bar{S}$ be two surfaces with unequal fundamental tensors $g_{\alpha\beta}$ and $\bar{g}_{\alpha\beta}$ respectively. If S and $\bar{S}$ are in geodesic correspondence and their total curvatures in corresponding points are equal to K, we have [2, p. 208]

$$\phi_{,\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} + K(g_{\alpha\beta} - \bar{g}_{\alpha\beta}) \tag{1}$$

where ϕ denotes the function $1/6 \log (\bar{g}/g)$. Finding the covariant derivative based upon the tensor $g_{\alpha\beta}$ we obtain

$$\phi_{,\alpha\beta\gamma} = \phi_{,\alpha\gamma}\phi_{,\beta} + \phi_{,\alpha}\phi_{,\beta\gamma} + (g_{\alpha\beta} - \bar{g}_{\alpha\beta})K_{,\gamma} - K\bar{g}_{\alpha\beta,\gamma}. \tag{2}$$

By the help of (1) and the relation [2, p. 208]

$$\bar{g}_{\alpha\beta,\gamma} = 2\bar{g}_{\alpha\beta}\phi_{,\gamma} + \bar{g}_{\beta\gamma}\phi_{,\alpha} + \bar{g}_{\alpha\gamma}\phi_{,\beta}$$

we reduce (2) into

$$\begin{aligned}\phi_{,\alpha\beta\gamma} = &- 4\phi_{,\alpha}\phi_{,\beta}\phi_{,\gamma} + 2(\phi_{,\alpha}\phi_{,\beta\gamma} + \phi_{,\beta}\phi_{,\gamma\alpha} + \phi_{,\gamma}\phi_{,\alpha\beta}) \\ &- K(g_{\beta\gamma}\phi_{,\alpha} + g_{\alpha\gamma}\phi_{,\beta} + 2g_{\alpha\beta}\phi_{,\gamma}) + (g_{\alpha\beta} - \bar{g}_{\alpha\beta})K_{,\gamma},\end{aligned} \tag{3}$$

from which it follows that

$$\phi_{,\alpha\beta\gamma} - \phi_{,\alpha\gamma\beta} = K(g_{\alpha\gamma}\phi_{,\beta} - g_{\alpha\beta}\phi_{,\gamma}) + K_{,\gamma}(g_{\alpha\beta} - \bar{g}_{\alpha\beta}) - K_{,\beta}(g_{\alpha\gamma} - \bar{g}_{\alpha\gamma}). \tag{4}$$

Since it is shown [2, p. 208] that

$$\phi_{,\alpha\beta\gamma} - \phi_{,\alpha\gamma\beta} = K(g_{\alpha\gamma}\phi_{,\beta} - g_{\alpha\beta}\phi_{,\gamma}),$$

it is obvious that (4) is equivalent to

$$K_{,\gamma}(g_{\alpha\beta} - \bar{g}_{\alpha\beta}) - K_{,\beta}(g_{\alpha\gamma} - \bar{g}_{\alpha\gamma}) = 0. \tag{5}$$

When the surfaces S and $\bar{S}$ are referred to orthogonal nets [2, p. 210], we have

$$\bar{g}_{\alpha\alpha} = \frac{g_{\alpha\alpha}}{U_\alpha^2 U_\beta}, \qquad \beta \neq \alpha,$$

where U_1 and U_2 are functions of u^1 and u^2 respectively. Then we have (5) reducible to

$$K_{,2}\, g_{11}\left(\frac{U_1^2U_2 - 1}{U_1^2U_2}\right) = 0, \qquad K_{,1}\, g_{22}\left(\frac{U_2^2U_1 - 1}{U_2^2U_1}\right) = 0,$$

which imply that $K_{,1}=0$ and $K_{,2}=0$ in the orthogonal coordinate system. Since $K_{,\alpha}$ is a tensor which vanishes in one coordinate system, it will vanish in all coordinate systems. Hence K is a constant. Therefore the surfaces S and $\overline{S}$ are applicable.

References

1. W. C. Graustein, Differential Geometry, The Macmillan Co., 1947.
2. L. P. Eisenhart, An Introduction to Differential Geometry with Use of the Tensor Calculus, Princeton University Press, 1947.

ON BOSE NUMBERS

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1. R. C. Das has established* a method for writing down the repetend of the recurring decimal for $1/n$, n being a positive integer prime to 10. In this connection, two numbers b (Bose number) and e (end number) were defined. These are the least positive integers that satisfy the relation

$$n \cdot e = 10 \cdot b - 1. \tag{A}$$

Let $1/n = .\dot{a}_1 a_2 \cdots \dot{a}_k$, so that

$$(10^k - 1)/n = [a_1 a_2 \cdots a_k] \tag{B}$$

where $[a_1 a_2 \cdots a_k]$, with a_i integers and $0 \leqq a_i \leqq 9$ $(i = 1, \cdots, k)$, stands for the expression $10^{k-1}a_1 + 10^{k-2}a_2 + \cdots + a_k$. Das has shown that from a knowledge of b and e only, the repetend $[a_1 a_2 \cdots a_k]$ can be written down. In the present note it is shown that b and e can also be used in a different manner to write down the repetend.

2. The present method consists of dividing the end number e by b, giving rise to a quotient e_1 and a remainder r_1. Then e_1 is placed after r_1, and the number $10r_1 + e_1$ so formed is again divided by b, giving rise to a quotient e_2 and a remainder r_2. The process is repeated. Our object is to prove the

THEOREM (i) *Continuation after k steps of the above process will cause the quotients $e_1, e_2, \cdots, e_k$ $(0 \leqq e_i \leqq 9)$ to repeat in the same order; and* (ii) $[e_1 e_2 \cdots e_k]$ *is the repetend of the recurring decimal for* $1/n$, *i.e.* $[e_1 e_2 \cdots e_k] = [a_1 a_2 \cdots a_k]$.

Proof: From the method of formation of the quotients $e_1, e_2, \cdots, e_k$ and the remainders $r_1, r_2, \cdots, r_k$, we can immediately write down the following equations

* This MONTHLY, vol. 56, No. 2. 1949, pp. 87–89.

$$\text{(C)} \qquad \begin{cases} e = be_1 + r_1 \\ 10r_1 + e_1 = be_2 + r_2 \\ \cdots\cdots\cdots\cdots \\ 10r_{k-1} + e_{k-1} = be_k + r_k. \end{cases}$$

Multiplying the k equations (C) by $10^k, 10^{k-1}, \cdots, 10$ successively and adding, we get the equation (D), after simplifying with the help of relation (A);

$$\text{(D)} \qquad (10^k - 1)e - ne[e_1e_2 \cdots e_k] = (10r_k + e_k) - e.$$

The equation (D) shows that e divides $10r_k+e_k$. Let $pe=10r_k+e_k$. Now dividing the equation (D) throughout by ne and using the relation (B) we get

$$\text{(E)} \qquad [a_1a_2 \cdots a_k] - [e_1e_2 \cdots e_k] = (p-1)/n.$$

It can be easily seen that $10r_k+e_k \leqq 10b-1$, which implies $p \leqq n$.* Relation (E) therefore is an equality between an integer and a proper fraction, which is absurd unless $p=1$ and, consequently, $[e_1e_2 \cdots e_k] = [a_1a_2 \cdots a_k]$. The latter relation establishes the second part of the proposition and $p=1$ implies that $r_k=0$ and $e_k=e$. As a consequence of these values of r_k and e_k, it is obvious that the process repeats itself after the kth step. Hence the proposition is established.

Example: Take $n=123$. Then $b=37$ and $e=3$. Dividing 3 by 37 and continuing the process as indicated above, we have

```
37) 3      ( 0 0 8 1 3
    0
    --
    30
     0
    --
    300
    296
    ---
     48
     37
    ---
    111
    111
    ---
     03 (The process repeats after this step.)
```

Hence

$$\frac{1}{123} = .\dot{0}0813\dot{}.$$

* From the inequalities $1 \leqq e \leqq 9$, $1 \leqq b \leqq n-1$, and $0 \leqq r_i \leqq b-1$ (all i) it is not hard to show first that $e_1 \leqq e \leqq 9$, and then that if $e_i \leqq 9$ then $10r_i+e_i = ne_{i+1}+r_{i+1} \leqq 10b-1$. It thus follows by induction that $0 \leqq e_i \leqq 9$ and hence moreover, for $i=k$, that $pe=10r_k+e_k \leqq 10b-1=ne$. (Referee.)

3. The method can be extended immediately to get the repetend of the recurring decimal for m/n, for any $m<n$. Suppose $m/n=.\dot{b}_1b_2\cdots\dot{b}_t$ which is equal to $[b_1b_2\cdots b_t]/(10^t-1)$. We shall show that the repetend $[b_1b_2\cdots b_t]$ can be obtained if we take the same b and e, and instead of dividing b into e, if we divide b into me and proceed in exactly the same manner as in the case of finding the repetend of the recurring decimal for $1/n$.

If, in the process of division of me by b, the successive quotient and remainders be $c_1, c_2, \cdots, c_t$ and $R_1, R_2, \cdots, R_t$, respectively, then we have the following equations (C′) corresponding to equations (C) above.

$$\text{(C')}\qquad \begin{cases} me = bc_1 + R_1 \\ 10R_1 + c_1 = bc_2 + R_2 \\ \cdots\cdots\cdots\cdots \\ 10R_{t-1} + c_{t-1} = bc_t + R_t. \end{cases}$$

Following steps similar to those in the previous proof, we get, corresponding to equation (D), the relation

$$\text{(D')}\qquad (10^t - 1)me - ne[c_1c_2\cdots c_t] = 10R_t + c_t - me.$$

Putting $10R_t+c_t=qe$, we get, corresponding to the relation (E),

$$\text{(E')}\qquad [b_1b_2\cdots b_t] - [c_1c_2\cdots c_t] = (q-m)/n.$$

By precisely the same arguments as before, this gives $q=m$, and $[c_1c_2\cdots c_t] = [b_1b_2\cdots b_t]$. The relation $q=m$ gives $10R_t+c_t=me$, which will mean that the quotients $c_1, c_2, \cdots$ repeat after the tth term. The proposition is therefore established.

Examples: (i) Take $m/n=25/123$. Dividing $b=37$ into $me=75$, we get,

```
37) 75      ( 2 0 3 2 5
    74
    --
     12
      0
     --
     120
     111
     ---
      93
      74
      --
      192
      185
      ---
        75  (The process repeats after this step.)
```

Hence the repetend is 20325 and $25/123=.\dot{2}0325\dot{}$.

(ii) Take $m/n = 41/123 \quad (=1/3).$

Dividing $b=37$ into $me=123$, we get,

$$\begin{array}{r} 37)\ 123\ (\ 3 \\ \underline{111} \\ 123 \end{array}$$

(The process repeats after this step.).

Hence the repetend is 3. So $41/123 = .\dot{3}$.

A NOTE ON NUMBER THEORY*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Introduction. Legendre [1] mentions a "very ingenious" proof by Sophie Germain that the Fermat equation

$$a^m + b^m + c^m = 0$$

has no integral solution in which $2m+1$ is prime and divides none of the numbers a, b, and c. He also considered the case in which m divides at least one of the numbers a, b, c whereas Germain [2] had assumed that none of the three is a multiple of m. It is very easy to remove the restriction on whether m divides one or more of the numbers a, b, c in the special case which follows.

2. THEOREM: *If $2m+1 (m>2)$ is a prime, there exists no integral solution of the equation*

$$a^m + b^m + c^m = 0$$

in which $2m+1$ fails to divide one of the integers a, b, c.

If a is relatively prime to $2m+1$, Fermat's theorem states that

$$a^{2m} - 1 \equiv (a^m - 1)(a^m + 1) \equiv 0 \pmod{2m+1}$$

since $2m+1$ is prime by hypothesis. Moreover, either a^m-1 or a^m+1, but not both, is divisible by $2m+1$ since their greatest common divisor cannot exceed 2. Hence, using analogous arguments in the case of b and c,

$$(1) \qquad a^m \equiv \pm 1, \qquad b^m \equiv \pm 1, \qquad c^m \equiv \pm 1 \pmod{2m+1},$$

thus proving the theorem.

COROLLARY: *If $4m+1$ is a prime, there exists no integral solution of the equation*

$$a^m + b^m + c^m = 0$$

in which $4m+1$ fails to divide one of the integers a, b, c, $a^{2m}+1$, $b^{2m}+1$, $c^{2m}+1$.

* Translated from the French by R. L. Wilson.

If $a \not\equiv 0 \pmod{4m+1}$, $a^{4m}-1 \equiv (a^{2m}-1)(a^{2m}+1) \equiv 0 \pmod{4m+1}$. Since $a^{2m}+1 \not\equiv 0 \pmod{4m+1}$, $a^{2m}-1 \equiv (a^m-1)(a^m+1) \equiv 0 \pmod{4m+1}$. Thus $a^m \equiv \pm 1 \pmod{4m+1}$ and the proof proceeds as before.

3. THEOREM: *If $2m+1$ is a prime and k is a positive integer, there exists no integral solution of the equation*

$$[k(2m+1)-1]a^m = b^m + c^m \tag{2}$$

in which $2m+1$ fails to divide one of the integers a, b, c.

Assume that none of the numbers a, b, c is divisible by $2m+1$. By Theorem 1, $a^m \equiv \pm 1 \pmod{2m+1}$ and hence $[k(2m+1)-1]a^m \equiv \pm 1 \pmod{2m+1}$. On the other hand $b^m \equiv \pm 1$, $c^m \equiv \pm 1 \pmod{2m+1}$, whence either $b^m+c^m \equiv 0$ or $b^m+c^m \equiv \pm 2 \pmod{2m+1}$. Therefore, at least one of the integers a, b, c must be a multiple of $2m+1$.

Example. If $m=3$, $k=2$, equation (2) becomes

$$13a^3 = b^3 + c^3.$$

This possesses at least one integral solution, namely $a=3$, $b=7$, $c=2$.

References

1. Legendre, *Essai sur la Theorie des Nombres*, (Second Supplément), *Sphinx-Oedipe*, 1909, pp. 97–128.

2. Cf. also Bachmann, *Niedere Zahlentheorie*, II, 1910, p. 475.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

TRANSFORMATIONS OF THE LAPLACIAN

R. P. AGNEW, Cornell University

It is the purpose of this note to make some simple remarks about the familiar tedious procedure involved in using a transformation of coordinates to convert the Laplacian operator from rectangular to spherical coordinates. Let the rectangular, cylindrical, and spherical coordinates of a point be, respectively, (x, y, z), (ρ, ϕ, z), and (r, ϕ, θ). The problem is to start with the Laplacian Δu defined by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{1}$$

and to show that

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}. \tag{2}$$

Direct use of the formulas

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta \tag{3}$$

to convert (1) into (2) is the familiar and tedious procedure noted above. It is very much easier to use the simpler formulas

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z \tag{4}$$

relating rectangular and cylindrical coordinates to convert (1) into the expression

$$\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} \tag{5}$$

in cylindrical coordinates.

When (5) has been obtained, the formulas

$$z = r\cos\theta, \qquad \rho = r\sin\theta, \qquad \phi = \phi \tag{6}$$

relating cylindrical and spherical coordinates may be used to convert (5) into (2); but a simple observation eliminates nearly all of the work involved in this step. We note that, except for the names of the variables involved, the transformation (6) is identical with the transformation (4). Since (4) converts the sum of the first two terms of the right member of (1) into the sum of the first three terms in the right member of (5), it follows that (6) converts the sum of the first and last terms of the right member of (5) into

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{7}$$

After obtaining the relatively much simpler fact that (6) converts the sum of the two central terms of the right member of (5) into

$$\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}, \tag{8}$$

we see without laborious calculation that (6) converts (5) into (2). Thus we are able to obtain both (5) and the more complicated (2) with a total amount of labor very little greater than that required to obtain (5) alone.

Since derivations of (2) appear in so many books and lectures in pure and applied mathematics, it would seem to be impossible to be certain that the above remarks are new. In any case, two facts indicate that the remarks have never

been adequately publicized. In the first place, (2) is very often derived by tedious use of (3). In the second place, there have been times when (5) was derived by use of (4), but the derivation of (2) was omitted on the ground that the derivation was too tedious.

Finally, we may ask whether there are other occasions in which one can make significant use of the fact that, except for the names of the variables involved, the transformations (4) and (6) are identical.

THE BERNOULLI DIFFERENTIAL EQUATION

A. BUCKLEY, Huddersfield Technical College

In an interesting article on the differential equation $(dy/dx)+Py=Qy^n$, under Classroom Notes (this MONTHLY vol. 59, 1952, pp. 632–633), M. R. Spiegel discusses solutions in the particular cases $n=0$ and 1. The same conclusions are evident if the equation is solved by the following method.

Multiply throughout the equation by an integrating factor I of the left-hand side, to give

$$I\left(\frac{dy}{dx}+Py\right)=Qy^n\frac{I^n}{I^{n-1}},$$

which reduces to

$$\frac{d(Iy)}{(Iy)^n}=\frac{Q}{I^{n-1}}\,dx.$$

The advantage of this form is that it holds for $n=0$ and 1 and only in the latter case does the easily derived solution differ from that obtained for all other values of n.

This method of solution was first published by Professor Neville in the Mathematical Gazette (vol. XVIII, 1934, p. 321) and was referred to by the present writer in a note to the same journal (vol. XXXIV, 1950, p. 304).

A MODIFICATION OF THE SIEVE OF ERATOSTHENES

V. C. HARRIS, San Diego State College

The following modification of the method known as the sieve of Eratosthenes may be of interest to classes studying elementary number theory.

Write the integers $1, 2, \cdots, n$ in natural order. Cross out every integer. Cross out every second integer, beginning with 2. Cross out every third integer, beginning with 3. Continue similarly for $4, 5, \cdots, n$. We find

(a) each unit has been crossed out once (there is just one unit, namely, 1),

(b) each prime has been crossed out twice, and the number of these is $\pi(n)$, the number of primes $\leqq n$,

(c) each composite number m, $1\leqq m\leqq n$, has been crossed out a number of times equal to $\tau(m)$, the number of divisors of m.

Centering our attention on a particular integer, say m, $1 \leqq m \leqq n$, we can write down the integer begun with each time that m is crossed out. Then

(d) the sum of these is $\sigma(m)$, the sum of the divisors of m,

(e) of these, the integers crossed out twice are the distinct prime divisors of m.

If, however, the process as described is carried out not for all of $1, 2, \cdots, n$ but only for the distinct prime divisors of n, then

(f) the number of integers not crossed out is $\phi(n)$, the number of (positive) integers $\leqq n$ and relatively prime to n.

THE RATE OF INTEREST IN INSTALLMENT PAYMENT PLANS

H. E. STELSON, Michigan State College

The purpose of this paper is to develop by a new method the formulas which are now in current use, to compare them to compound interest, and to present some new formulas.

1. Historical approach. The formulas which are currently presented in texts are known as the Constant Ratio, Series of Payments and Interest at End formulas. The formulas have been developed independently from different assumptions.

The Constant Ratio formula is derived on the assumption that each payment is composed of a principal repayment and an interest repayment in the same ratio that the original unpaid balance is to the interest. The basic assumption for the Series of Payments formula is that the sum of the series of payments is the outstanding debt at the beginning of the installment term. The Interest at End or Residuary formula* is derived on the assumption that the payments should be used to repay the principal first, then after the principal has been repaid, to pay the interest.

Formulas for the three methods are obtained by considering the principal outstanding during each period, finding the sum of all these principals (an arithmetic progression) and substituting this total principal, considered to be in use for one payment period, in the simple interest formula.

2. Alternate derivation—new method. The values of different obligations can be compared only if they are accumulated or discounted to the same date. That is, the value of the debt on a certain date must equal the value of the payments. The date chosen for comparison is called the comparison or focal date.

We now make the following assumption, which will be used to derive the three formulas already mentioned. For all payments (or debts) made before the focal date, the accumulation factor $(1+nr)$ is used, while for all payments made after the focal date, the discount factor $(1-nr)$ is used.

The following symbols are used:

* This assumption is equivalent to the Merchant's Rule.

R = periodic payment,
r = periodic rate,
n = number of periodic payments (not counting the down payment),
B = unpaid balance at the beginning of the credit period (cash price less the down payment, if any),
I = total carrying charge or cost of the loan, or
$I = Rn - B$.

As an aid in considering the focal date, we present a line diagram for an installment payment plan.

Payments		R	$R \cdots R$
	|———	|———	|——— |
Time	0	1	$2 \cdots n$.

(A) If the focal date is taken at n in the line diagram (the end of the payment period), we use the Merchant's rule to obtain the equation,

$$B(1 + nr) = R[1 + (n - 1)r] + R[1 + (n - 2)r] + \cdots + R(1 + r) + R.$$

Solving this equation for r gives the Interest at End (or the Residuary) formula,

$$r_I = \frac{2I}{n(B + R - I)} = \frac{2I}{B(n + 1) - I(n - 1)}.$$

(B) If the focal date is taken at the average time, $(n+1)/2$, we obtain the Constant Ratio formula,

$$r_c = \frac{2I}{B(n + 1)}.$$

(C) If the focal date is taken at 0 (the beginning of the payment period), we have the Series of Payments formula,

$$r_s = \frac{2I}{Rn(n + 1)} = \frac{2I}{B(n + 1) + I(n + 1)}.$$

It can be shown by using the second formulas for r_I and r_s that the denominator for r is increased by $2I$ for each period the focal date is moved to the left in the line diagram. Hence

$$r_s < r_c < r_I.$$

3. A new formula—interest charged uniformly. In order to get a closer approximation to r, the compound interest rate, we move the focal date to $n/2$. This gives the very simple formula

$$r_B = \frac{2I}{n(B + R)}.$$

It can readily be proved that r_B is the harmonic mean of r_s and r_I.

In the derivation of the formulas for Constant Ratio, Series of Payments, and Interest at End, it is assumed that the charge for the loan, $I=Rn-B$, is made at the end of the period. It is more accurate to assume that the charge is paid uniformly throughout the payment interval or that it is paid at the middle of the payment interval. In the Constant Ratio method, the focal date is taken at time $(n+1)/2$. If the interest charge is paid at time $n/2$ then the debtor gets to hold the interest for $1/2$ period and the debt should be increased by $rI/2$. Hence using the focal date $(n+1)/2$ we get

$$B\left(1+\frac{n+1}{2}r\right)+\frac{Ir}{2}=R\left(1+\frac{n-1}{2}r\right)+\cdots+R\left(1+\frac{r}{2}\right)$$
$$+R\left(1-\frac{r}{2}\right)+R\left(1-\frac{3r}{2}\right)+\cdots+R\left(1-\frac{n-1}{2}r\right)$$

or

$$r=\frac{2I}{n(B+R)}=r_B.$$

If a similar correction is made in the same manner to the formulas for Interest at End and Series of Payments, they also reduce to the formula for r_B.

In order to compare the accuracy of r_B with r, we expand the formula for compound interest,

$$B = Ra_{\overline{n}|} \text{ in the series, } (Rn = B + I)$$

$$(1)\qquad B=\frac{2I}{r(n+1)}\left[1-\frac{(n-1)r}{6}+\frac{(n-1)(n-2)r^2}{36}+\cdots\right].$$

Also

$$(2)\qquad B=\frac{2I}{r_B(n+1)}\left[1-\frac{r_B}{2}\right].$$

Dividing (1) and (2) and solving for r_B-r we obtain the error,

$$r_B-r=\frac{n-4}{6}r^2-\frac{n-2}{4}r^3\cdots.$$

4. The direct ratio formula—Interest charged throughout the payment period in proportion to the loan outstanding. The direct ratio formula assumes that the total interest charge is distributed to each installment in proportion to the outstanding balance at that time. Under this assumption, we find the average time for the total loan. Since the principal outstanding for the first period is B, for the second period is $B-(B/n)$, *etc.*, the average time is

$$\frac{B\cdot 1+\left(B-\frac{B}{n}\right)\cdot 2+\left(B-\frac{2B}{n}\right)\cdot 3\cdots+\left[B-\frac{(n-1)B}{n}\right]n}{B\quad+\left(B-\frac{B}{n}\right)\quad+\left(B-\frac{2B}{n}\right)\quad\cdots+\left[B-\frac{(n-1)B}{n}\right]}=\frac{n+2}{3}.$$

Placing the focal date at $(n+2)/3$ and using the line diagram, we obtain in the same way that the formulas for r_s, r_c, and r_I were derived, a formula for the rate

$$r_a=\frac{6I}{3B(n+1)+I(n-1)}.$$

This formula gives results remarkably close to the compound interest rate.

The error is given by

$$r-r_a<\frac{(n-1)(n+2)r^3}{36}.$$

The formula for r_a assumes that all of the payments are equal. However, many installment plans have an irregular last payment. Again, we assume the focal date at $(n+2)/3$, and using the line diagram in the same manner as in the derivation of r_a we obtain the formula

$$r_a'=\frac{2I}{n(B+Z)+(I/3)(n-4)}$$

where Z is the last irregular payment.

References

1. Mergendahl and Foster, One Hundred Problems in Consumer Credit, Pollak Foundation, Newton, Mass.

2. H. E. Stelson, The rate of interest in installment payment plans, this MONTHLY, vol. 56, 1949, pp. 257–261.

3. Neifeld, M. R. Neifeld's Guide to Installment Computations, Mack Publishing Company, Easton, Pennsylvania, 1951.

4. Rider and Fischer, Mathematics of Investment, Rinehart & Co., 1951, pp. 19–22.

MORE ON TAYLOR'S THEOREM IN A FIRST COURSE

C. P. NICHOLAS, U.S. Military Academy

This MONTHLY (vol. 58, 1951, pp. 559–562) carried a derivation of Taylor's Theorem which I offered as suitable for simplifying the subject in a first course. The derivation can be further simplified, as shown below. Like the earlier derivation, the new one presupposes that the student can evaluate a successive integral; and also that he has been led into the problem sufficiently to know that we wish to find an approximate value of $f(x)$, based on known values of $f(x)$ and its first n derivatives at a neighboring point $x=a$.

The simplification occurs at the start where, instead of developing two special formulas for the value of an increment, we use the already familiar relationship:

$$\int_a^x f'(x)dx = f(x) - f(a).$$

This may be transposed to read

$$f(x) = f(a) + \int_a^x f'(x)dx. \tag{I}$$

Now, (I) may be used as an iteration formula to expand any function which has successive derivatives. Specifically, it may be used to expand the function under the integral sign of (I) itself. Thus, applying (I) to $f'(x)$ we have:

$$f(x) = f(a) + \int_a^x \left[f'(a) + \int_a^x f''(x)dx \right] dx$$

$$= f(a) + \int_a^x f'(a)dx + \int_a^x \int_a^x f''(x)(dx)^2.$$

Continuing, we apply (I) successively to $f''(x)$ under the double integral sign, then to $f'''(x)$ under the triple integral sign, and so on. Thus:

$$f(x) = f(a) + f'(a)\int_a^x dx + f''(a)\int_a^x \int_a^x (dx)^2 + f'''(a)\int_a^x \int_a^x \int_a^x (dx)^3$$

$$+ \cdots + f^{n-1}(a)\int_a^x \int_a^x \cdots \int_a^x (dx)^{n-1}$$

$$+ \int_a^x \int_a^x \int_a^x \cdots \int_a^x f^{(n)}(x)(dx)^n.$$

Evaluating each integral except the last, we obtain Taylor's formula:

$$f(x) = f(a) + f'(a)\cdot(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots$$

$$+ f^{(n-1)}(a)\frac{(x-a)^{n-1}}{(n-1)!} + R_n$$

where R_n, the remainder after n terms, is given by

$$R_n = \int_a^x \int_a^x \cdots \int_a^x f^{(n)}(x)(dx)^n.$$

The Lagrange form of the remainder may now be derived as explained in

my previous article. Professor H. J. Hamilton followed with an excellent derivation of the Cauchy form in this MONTHLY (vol. 59, 1952, p. 320).

It should be noted that the foundation of this proof includes the Fundamental Theorem of Integral Calculus, which in turn presupposes the Mean-Value Theorem for Derivatives, usually derived with the aid of Rolle's Theorem. Thus, the proof does not fundamentally avoid Rolle's Theorem. Its advantage lies in the fact that it avoids the elaborate auxiliary function usually introduced when Rolle's Theorem is employed to prove Taylor's Theorem or the Extended Theorem of the Mean.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1066. *Proposed by A. Zirakzadeh, Oklahoma A. and M. College*

Inscribe a trapezoid in a given quadrilateral such that the bases of the trapezoid will be parallel to one of the diagonals of the quadrilateral and the other two sides will pass, respectively, through two given points.

E 1067. *Proposed by Tatuya Sasakawa, Kyota University, Japan*

Show that

$$n! = \sum_{s=0}^{n} \sum_{i=0}^{s} (-1)^{s-i} {}_nC_s \, {}_sP_i.$$

E 1068. *Proposed by W. O. Pennell, Exeter, N. H.*

Given a triangle with sides a, b, c and $s^2 = 2ab$, where s is the semiperimeter. Show that: (1) $s < 2a$, $s < 2b$, (2) $a > c$, $b > c$.

E 1069. *Proposed by Robert Buehler, Arthur Gregory, and J. R. Wilson, Sandia Corporation, Albuquerque, N. M.*

How un-isosceles can a triangle be?

E 1070. *Proposed by A. W. Walker, University of Toronto*

Find the lowest order differential equation satisfied by all conics with a given eccentricity e.

SOLUTIONS

Jumping Off an Asteroid

E 1036 [1952, 633] *Proposed by W. R. Ransom, Tufts College*

How large an asteroid could a man jump clear off of?

Solution by W. M. Stone, Boeing Airplane Co., Seattle, Wash. Using the subscript e to refer to earth magnitudes, the acceleration due to gravity at the surface of any spherical asteroid of radius R and mass M would be

$$g = g_e MR_e^2/M_eR^2.$$

To jump a vertical height h on the earth's surface a man must acquire a velocity $v_e = (2g_eh)^{1/2}$. We equate this velocity to the velocity of escape from the asteroid, $v = (2gR)^{1/2}$:

$$2g_eh = 2g_eMR_e^2/M_eR = 2g_eCR^2/C_eR_e,$$

or

$$R^2 = (C_e/C)R_eh,$$

where C and C_e are the specific densities of the asteroid and earth respectively.

Also solved by Ray Aronson and H. S. Wilf (jointly), R. H. Boyer, Julian Braun, C. W. Bruce, A. M. Glicksman, H. W. Hickey, P. F. Hultquist, P. B. Johnson, M. S. Klamkin, C. S. Ogilvy, Azriel Rosenfeld, C. M. Sandwick, Sr., C. Swanson, and the proposer.

Glicksman considered also the interesting possibility where the man jumps horizontally, that is, tangentially. The man could then continue to move around the asteroid, like a satellite, just above its surface, and he could be said to have jumped "clear" of the asteroid. In this case one finds

$$R^2 = 2(C_e/C)R_eh,$$

whence "a man can jump completely *around* an asteroid whose radius is $\sqrt{2}$ times as large as that of the biggest asteroid he could jump completely *away* from."

Johnson emphasized the fact that h is the height a man can lift his center of gravity, and must not be confused with his high jump mark. In a good high jump the jumper's center of gravity starts off several feet above the ground and actually passes under the bar as the jumper goes over. It takes a good jumper to have an h of two feet.

Criterion for a Real Cubic to Have Pure Imaginary Roots

E 1037 [1952, 633]. *Proposed by A. G. Anderson, Oberlin College*

Show that a necessary and sufficient condition that a real cubic equation $ax^3+bx^2+cx+d=0$ have one real and two pure imaginary roots is that $ba=ad$ and $ac>0$.

Solution by Sidney Kravitz, Great Notch, N. J. The equation whose roots are A, $+Bi$, and $-Bi$, where A and $B\neq 0$ are real, is $x^3-Ax^2+B^2x-AB^2=0$, thus satisfying the conditions $bc=ad$ and $ac>0$. Conversely, if $bc=ad$ and $ac>0$, then the real cubic equation $ax^3+bx^2+cx+d=0$ has the real root $-b/a$, and the two pure imaginary roots $+\sqrt{c/a}\,i$ and $-\sqrt{c/a}\,i$.

Also solved by J. M. Anderson, J. W. Baldwin, A. P. Boblétt, Julian Braun, W. E. Briggs, C. S. Carlson, F. E. Cothran, Marian E. Daniells, I. A. Dodes, David Ellis, H. Emich, A. L. Epstein, H. M. Feldman, Calvin Foreman, Arthur Gregory, Bernard Greenspan, Emil Grosswald, S. W. Hahn, B. A. Hausmann, Vern Hoggatt, Douglas Holdridge, Willard James, W. C. Janes, John Jones, Jr., Ray Jurgensen, Aida Kalish, J. M. Kingston, M. S. Klamkin, J. D. E. Konhauser, A. E. Livingston, D. C. B. Marsh, B. Martin, Morris Morduchow, Leo Moser, Roberto Muguercia, C. S. Ogilvy, L. L. Pennisi, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, Sr., John Sawyer, A. Sisk, William Small, P. Somanadham and K. Subba Rao (jointly), C. R. Sparks, R. H. Sprague, W. M. Stone, C. Swanson, W. R. Talbot, W. C. Taylor, Jr., J. A. Tierney, Peter Treuenfels, R. Z. Vause, R. R. Williams, Jr., David Zeitlin, and the proposer. Late solutions by S. Parameswaran and Louisa S. Grenstein.

Pennisi pointed out that if the condition $bc=ad$ and $ac>0$ be changed to $bc=ad$ and $ac<0$ we get a necessary and sufficient condition that all three roots be real with two of them equal but of opposite sign.

James noted that since $bc=ad$ and $ac>0$ imply that a and c must have the same sign and that b and d must have the same sign we have an easy and quick check on the possibility of a real cubic having pure imaginary roots.

Stone called attention to von Karman and Biot, *Mathematical Methods in Engineering*, McGraw-Hill, New York (1940), p. 242.

Divisibility by 27

E 1038 [1952, 633]. *Proposed by S. B. Townes, University of Hawaii*

Show that if a number of $3n$ digits is divisible by 27, then any number with the same digits cyclically permuted will also be divisible by 27.

Solution by Leo Moser, University of Alberta. We shall prove the following more general result:

If N has sn digits when expressed in base r, then $t \mid N$, $t \mid (r^s-1)$, and $(r, t)=1$ imply t divides any number obtained from N by a cyclic permutation of its digits (in base r).

Proof. Let the last (on the right) digit of N be a. It clearly suffices to consider

the effect of putting this digit first. If the resulting number is M then

$$M = (N - a)/r + ar^{sn-1} = [N + a(r^{sn} - 1)]/r.$$

Now $t \mid (r^s - 1)$ and $(r^s - 1) \mid (r^{sn} - 1)$, so that $t \mid (r^{sn} - 1)$. Further, $t \mid N$ and $(r, t) = 1$. It follows that $t \mid M$, and the proof is complete.

The given problem is the case $r = 10$, $s = 3$, $t = 27$. The case $r = 10$, $s = 3$, $t = 37$ shows that 27 may be replaced by 37 in the given problem.

Also solved by Norman Anning, J. W. Baldwin, R. H. Boyer, Julian Braun, Monte Dernham, I. A. Dodes, A. L. Epstein, Lloyd Fulk, Bernard Greenspan, Arthur Gregory, S. W. Hahn, B. A. Hausmann, Vern Hoggatt, Douglas Holdridge, J. E. Householder, H. K. Humphrey, H. I. James, M. S. Klamkin, T. C. Littlejohn, A. E. Livingston, D. C. B. Marsh, L. A. Ringenberg, C. M. Sandwick, Sr., P. Somanadham and K. Subba Rao (jointly), R. H. Sprague, J. A. Tierney, R. Z. Vause, Leroy Warren, and the proposer. Late solutions by J. P. Jackson, Herb Lechner, Hilbert Levitz, R. J. Painter, S. Parameswaran, and R. H. Spencer.

Minimum Product of Perpendicular Diameters of an Ellipse

E 1039 [1952, 633]. *Proposed by I. W. Burr, Purdue University*

Minimize the product of two perpendicular central chords of a given ellipse.

Solution by Julian Braun, Washington, D. C. Employing usual notation, the polar equation of an ellipse with pole at the center is

$$r^2 = b^2/(1 - e^2 \cos^2 \theta).$$

Let

$$r_1 = b/(1 - e^2 \cos^2 \theta_1)^{1/2}.$$

Then the length of the semi-central chord perpendicular to r_1 is

$$r_2 = b/(1 - e^2 \sin^2 \theta_1)^{1/2}.$$

The product of the two perpendicular central chords is

$$4r_1r_2 = 4b^2/[1 - e^2 + (e^4/4) \sin^2 2\theta_1]^{1/2},$$

which is clearly a minimum for $\theta_1 = 45°$. Thus the required central chords make angles of 45° with the axes of the ellipse and their product is $4b^2/(1 - e^2/2)$ or $8a^2b^2/(a^2 + b^2)$.

Also solved by Leon Bankoff, A. P. Boblétt, W. E. Briggs, R. C. Clelland, I. A. Dodes, Bernard Greenspan, Arthur Gregory, Vern Hoggatt, Douglas Holdridge, R. Huck, Ray Jurgensen, M. S. Klamkin, D. M. Mandelbaum, Jerome Manheim, D. C. B. Marsh, George Millman, C. N. Mills, R. K. Morley, C. S. Ogilvy, L. A. Ringenberg, P. Somanadham and K. Subba Rao (jointly),

C. R. Sparks, C. Swanson, W. R. Talbot, J. A. Tierney, Peter Treuenfels, H. S. Wilf, and the proposer. Late solutions by H. H. Berry and A. E. Livingston.

Sparks pointed out the interesting fact that the minimum product of two perpendicular central chords of a given ellipse is the *harmonic* mean, whereas the maximum product is the *geometric* mean, of the squares of the axes of the ellipse.

Comparison of Merchant's Rule and Residuary Method

E 1040 [1952, 633]. *Proposed by H. E. Stelson, Michigan State College*

On page 20 of Rider and Fischer, *Mathematics of Investment*, is the statement, "It does not appear to have been generally recognized that the assumptions underlying the Residuary method are equivalent to those of the Merchant's Rule and hence the two methods yield identical results."

Prove that the Merchant's Rule and the Residuary (Interest at End) method give the same results for the case where the payments are all equal except the last and where the interest payment amounts to more than a single payment.

Solution by Julian Braun, Washington, D. C. Let A = initial unpaid balance, K = carrying charge, m = total number of payments per year, n = total number of payments in the contract, i = simple rate of interest, P = amount of each payment (except the last).

In the Residuary method the amount on which interest is computed for the jth period is $A-(j-1)P$, so that, according to this method,

$$K = (i/m)\sum_{j=1}^{n}[A - (j-1)P],$$

whence

$$i = 2mK/[2An - nP(n-1)].$$

In the Merchant's Rule the principal earns interest to the final date. The jth payment ($j<n$) earns interest for $n-j$ periods, and the last payment is $(A+K)-(n-1)P$. Thus, for this method,

$$A(1 + ni/m) = P\sum_{j=1}^{n-1}[1 + (n-j)i/m] + (A + K) - (n-1)P.$$

Solving this for i yields the same result as above.

Also solved by the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4526 [1953, 123] corrected. *Proposed by Peter Ungar, New York University*

Show that for any $k>1$ there exist k-chromatic graphs which contain no three mutually connected nodes. (If it requires k colors to color the nodes so that no two nodes of the same color are connected by an edge, the graph is k-chromatic.)

4538. *Proposed by Albert Wilansky, Lehigh University*

Given $\sum|b_k|<\infty$, must there exist a constant M such that whenever $\{x_n\}$, $n=1, 2, \cdots$, is a convergent sequence satisfying

$$\left|\sum_{k=1}^{n-1} b_k x_k + x_{n-1} + x_n\right| < 1$$

for all n, then $|\lim x_n| < M$?

4539. *Proposed by J. Gallego-Diaz, Madrid, Spain*

Determine

$$\lim_{n\to\infty} [\tanh 1 + \tanh 2 + \cdots + \tanh n - \log\cosh n].$$

4540. *Proposed by J. Gallego-Diaz, Madrid, Spain*

Determine the equation of the most general curve such that the locus of the centers of equilateral triangles inscribed in it is the same curve. (The equilateral hyperbola is a particular case.)

4541. *Proposed by Jack Warga, Republic Aviation Corporation, Farmingdale, N. Y.*

Let $d_k(n)$ be defined for integers $k\geq -1$ and $n\geq 1$ by:

$$d_{-1}(n) = \delta_{1n} = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1, \end{cases}$$

$$d_{k-1}(n) = \sum_{r_1 r_2 \cdots r_k = n} 1 = \text{the number of}$$

representations of n as a product of k factors, for $k\geq 1$.

Let $\mu(j)$ be the Möbius function.

Let $\lambda(j)=\alpha_1+\alpha_2+\cdots$ for $j=p_1^{\alpha_1}p_2^{\alpha_2}\cdots$, (the decomposition of j into prime powers).

Prove that, for all integers j and t such that $t>\lambda(j)$,

$$\mu(j) = -\sum_{k=1}^{t}(-1)^k\binom{t}{k}d_{k-2}(j).$$

4542. *Proposed by C. D. Olds, San Jose State College, Calif*

The n points (x_i, y_i), $i=1, 2, \cdots, n$, should lie on a circle but they fail to do so. What circle shall we take which most nearly fulfills the condition that $\sum d_i^2$ is a minimum where

$$d_i = [(h-x_i)^2+(k-y_i)^2]^{1/2}-r?$$

The origin of coördinates is at the centroid of the given points; and (h, k) is the center, r the radius, of the circle to be found. If necessary, assume that d_i is small compared with r.

SOLUTIONS

Area of the Morley Triangle of a Triangle

4477 [1952, 110]. *Proposed by C. E. Springer, University of Oklahoma*

Given a triangle with angles α, β, γ. Show that the ratio of the area of the Morley triangle of the given triangle to the area of the given triangle is

$$\frac{\left(\prod \sin^2\frac{\alpha}{3}\right)\left(1-4\sum\cos^2\frac{\alpha}{3}+16\prod\cos\frac{\alpha}{3}\right)}{(\prod\sin\alpha)\left(\prod\sin\frac{\pi-\alpha}{3}\right)}.$$

(If the angles of a triangle be trisected, the intersections of the pair of trisectors adjacent to each side determine an equilateral triangle called the Morley triangle of the given triangle. See also [1943, 552].)

Solution by Roscoe Woods, The State University of Iowa. Let the given triangle ABC be chosen as the reference triangle of a system of trilinear normal coordinates. Then it is known that the area K of any triangle $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ is given by the formula $2K=(R/S)(x_1y_2z_3)$, where R and S are respectively the circumradius and area of the triangle ABC, and where the symbol $(x_1y_2z_3)$ is a familiar determinant of the third order with the elements x_1, y_2, z_3 along its principal diagonal. In the application of this formula the coördinates x_i, y_i, z_i must be the actual coördinates of the points $P_i(i=1, 2, 3)$.

Let P_1 be the vertex of the Morley triangle formed by the intersections of the trisectors adjacent to the side a of the triangle ABC. The actual coördinates of the vertex P_1 are then readily found by the use of trigonometry to be

$$\left\{ \frac{a \sin \frac{\beta}{3} \sin \frac{\gamma}{3}}{\sin\left(\frac{\beta+\gamma}{3}\right)}, \frac{2a \sin \frac{\beta}{3} \sin \frac{\gamma}{3} \cos \frac{\gamma}{3}}{\sin\left(\frac{\beta+\gamma}{3}\right)}, \frac{2a \sin \frac{\beta}{3} \sin \frac{\gamma}{3} \cos \frac{\beta}{3}}{\sin\left(\frac{\beta+\gamma}{3}\right)} \right\}.$$

The coördinates of the other two vertices P_2 and P_3 of the Morley triangle may be written down by permuting the symbols a, b, c and α, β, γ. The area K of the Morley triangle is then readily found by the above formula to be

$$(1) \qquad 2K = \frac{Rabc\left(\prod \sin^2 \frac{\alpha}{3}\right)\left(1 - 4\sum \cos^2 \frac{\alpha}{3} + 16 \prod \cos \frac{\alpha}{3}\right)}{S\left(\prod \sin \frac{\pi - \alpha}{3}\right)}.$$

From the relations $4RS=abc$ and $S=2R^2 \sin \alpha \sin \beta \sin \gamma$, equation (1) reduces at once to the proposed form.

Also solved by Mrs. R. A. Hudson and the Proposer.

Basis in a Banach Space

4478 [1952, 186]. *Proposed by Albert Wilansky, Lehigh University*

A basis in a Banach space B is a set of elements $\{x_n\}$ such that every element of the space is a unique, finite or infinite, linear combination of elements of the set.

(1) Prove that the x_n are isolated.

(2) If $\sum x_n$ is an element of B, show that there is a divergent sequence $\{s_n\}$ of numbers such that $\sum s_n x_n$ is an element of B.

Solution by G. G. Lorentz, University of Toronto, Canada. (1) If x_p is not isolated, then

$$x_p = \lim x_{n_k}, \qquad p < n_1 < n_2 < \cdots.$$

In this case,

$$x_p = x_p,$$

$$x_p = x_{n_1} + (x_{n_2} - x_{n_1}) + (x_{n_3} - x_{n_2}) + \cdots + (x_{n_k} - x_{n_{k-1}}) + \cdots$$

are two different representations of x_p, which is a contradiction.

(2) The hypothesis that the x_n form a basis is not necessary here. Let $\sum x_n$ converge. Then $\|x_n\| \to 0$ and there is a sequence $n_1 < n_2 < \cdots$ such that $\|x_{n_k}\| \leqq k^{-3}$. Putting $s_{n_k} = k$, $k = 1, 2, \cdots$, $s_n = 1$ for $n \neq n_k$, we see that $\sum s_n x_n$ is convergent, while s_n diverges.

Also solved by Abraham Charnes, R. C. James, Solomon Leader, Karl Zeller, and the Proposer.

An Infinite Set a_n such that $\sum a_n^p=0$ for all p

4479 [1952, 187]. *Proposed by D. J. Newman, Harvard University*

Suppose that all the series

$$\sum_{n=0}^{\infty} \alpha_n, \quad \sum_{n=0}^{\infty} \alpha_n^2, \quad \sum_{n=0}^{\infty} \alpha_n^3, \cdots$$

converge to zero, where the α_n are complex numbers. Does it follow that all the $\alpha_n=0$?

Solution by Karl Zeller, Tübingen, Germany. Put $a=\{\alpha_k\}$, $k=1, 2, \cdots$; $\theta a=\{\theta\alpha_k\}$ for any constant θ; $\phi_p(a)=\sum_{k=0}^{\infty} \alpha_k^p$, whence $\phi_p(\theta a)=\theta^p \cdot \phi_p(a)$. If $b=\{\beta_0, \cdots, \beta_q \neq 0, 0, 0, \cdots\}$ and $c=\{\gamma_0, \cdots, \gamma_j \neq 0, 0, 0, \cdots\}$, let $b \circ c=\{\beta_0, \cdots, \beta_q, \gamma_0, \cdots, \gamma_j, 0, 0, \cdots\}$.

A sequence $a \neq \{0, 0, \cdots\}$ can be exhibited which has the property

$$\phi_p(a) = 0, \qquad p = 1, 2, \cdots.$$

Let a be the sequence whose terms are the non-zero terms of the sequences a_n determined inductively as follows:

$$a_1 = \{1, -1, 0, 0, \cdots\},$$

$$a_n = a_{n-1} \circ \frac{\rho_n}{n} a_{n-1} \circ \cdots \circ \frac{\rho_n}{n} a_{n-1},$$

in which $|\rho_n| = 1$, $\rho_n^n = -1$, and the set $(\rho_n/n)a_{n-1}$ occurs n^n times. It is then true that

$$\phi_n(a_n) = \phi_n(a_{n-1}) + n^n \frac{\rho_n^n}{n^n} \phi_n(a_{n-1}) = 0.$$

Thus, for every $p=1, 2, \cdots$, we have a relation of the form

$$a = a_p \circ \lambda_2 a_p \circ \lambda_3 a_p \circ \cdots$$

with $\lambda_j \to 0$, from which the desired result follows.

Also solved by Robert Frucht, Fritz Herzog, E. C. Straus, and the Proposer, all of whom note that if $\sum \alpha_n^p$ is absolutely convergent for all p, then all $\alpha_n=0$.

A Three Dimensional Cantor Set

4480 [1952, 187]. *Proposed by Rufus Isaacs, the Rand Corporation, Santa Monica, Calif.*

Given a closed solid polyhedron with faces which are equilateral triangles, we can construct another by the following process. Join the mid-edge points of each face dividing it into four triangles. On each face place externally a regular tetrahedron having the central triangle for a base.

If we start with a regular tetrahedron and repeat this process, show that the limiting figure is a cube less a certain point set which is non-enumerable yet of measure zero.

Solution by E. M. Zaustinsky, University of Southern Calif. Let the length of an edge of the given tetrahedron be 1. If at the nth step of the construction the figure has M faces each with edge s, at the $(n+1)$th step it will have $6M$ faces each with edge $s/2$. Now the $(n+1)$th step adds the volume of M tetrahedrons with edge $s/2$. We note that the vertices of the original tetrahedron and the new vertices produced by the first step are the eight vertices of a cube of edge $\sqrt{2}/2$. Since the volume of a regular tetrahedron with edge s is $V = s^3\sqrt{2}/12$, the sum of the volumes of the tetrahedrons up to the nth step is

$$V_n = \frac{\sqrt{2}}{12}\left\{1 + \frac{4}{8} + \frac{4\cdot 6}{8^2} + \cdots + \frac{4\cdot 6^{n-1}}{8^n}\right\}.$$

The volume of the cube is $\sqrt{2}/4$. We regard the tetrahedrons as being subtracted from the cube during the construction. The complement, then, of the sum of the tetrahedrons with respect to the cube is a Borel set, is measurable, and has measure at the nth step equal to

$$\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{12}\left\{1 + \frac{1}{2}\sum_{i=0}^{n-1}\left(\frac{6}{8}\right)^i\right\},$$

whence the measure of the residual set when $n \to \infty$ is 0.

Now let us consider that subset of the residual set which lies in one edge of the cube. At the nth step of the construction we removed only a finite number of points from this subset and therefore only a countable number have been removed from the limiting set. The points in the residual set along the edge are therefore uncountable and, *a fortiori*, so are the points of the entire residual set.

Also solved by the Proposer.

Cubes with Same Unit's Digit

4481 [1952, 187]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a system of numeration with base B, there are n one-digit numbers less than B whose cubes have $B-1$ as the unit's digit. Determine the relation between B and n.

Solution by Emil Grosswald, Institute for Advanced Study. Let

$$B = 3^a p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} q_1^{b_1} \cdots q_t^{b_t}$$

where p_i, q_j are primes, $p_i \equiv 1 \pmod 3$, $q_j \equiv 2 \pmod 3$. We will show that

(1) $$n = 3^{r+s} \qquad s = \begin{cases} 0, & a \leqq 1, \\ 1, & a \geqq 2. \end{cases}$$

Let n' be the number of one-digit numbers whose cubes have 1 as the unit's digit. We first show that $n=n'$ and then determine the value of n'. Let $N=B-m(1\leqq m\leqq B-1)$; then the identity $N^3=B(B^2-3mB+3m^2-c)+(cB-m^3)$ holds for every c. If N is a number which satisfies the condition of the problem, we obtain

$$cB - m^3 = B - 1, \qquad B^2 - 3mB + 3m^2 - c \geqq 0$$

or the equivalent system

(2) $$B(c-1) = m^3 - 1, \qquad 1 \leqq m \leqq B - (B-1)^{1/3}.$$

Every solution of the diophantine system (2) in integers c, m leads to an integer $N=B-m$ satisfying the condition of the problem, and conversely; therefore n equals the number of solutions of (2). We observe that this number does not change if we replace the second inequality by $m\leqq B$. Indeed, let $\mu=[(B-1)^{1/3}]$, $m=B-\nu$ with $0\leqq\nu\leqq\mu\leqq(B-1)^{1/3}$. Then $m^3-1=(B-\nu)^3-1 \equiv B-\nu^3-1 \pmod B$ so that B is not a divisor of (m^3-1) if $0\leqq\nu<(B-1)^{1/3}$. It follows that n is the number of solutions of the system

(3) $$m^3 \equiv 1 \pmod B, \qquad 1 \leqq m \leqq B,$$

and this equals n'.

(3) is always satisfied by $m=1$, so that to every base B, $N^3=(B-1)^3$ has $B-1$ as the unit's digit. For $m>1$ it follows from the Fermat-Euler theorem (H & W, th. 88)* that (3) is possible only if 3 is a divisor of $\phi(B)$, where $\phi(x)$ is the Euler function (totient).

We can now complete the proof of (1). $4(x^2+x+1)\equiv 0 \pmod p$ is equivalent to $(2x+1)^2\equiv -3 \pmod p$ and has two distinct solutions when -3 is a quadratic residue (mod p). Since $p\equiv 1 \pmod 3$ this condition is met (H & W, th. 96). Consequently

$$m^3 - 1 = (m-1)(m^2 + m + 1) \equiv 0 \pmod p$$

has $m_0=1$ and two other solutions, all distinct (mod p). The same holds (H & W, th. 123) for $m^3-1\equiv 0 \pmod{p^a}$. If $q\equiv 2 \pmod 3$, $(2x+1)^2\equiv -3 \pmod q$ has no solutions and $m^3-1\equiv 0 \pmod{q^b}$ has the unique solution $m_0=1$. Finally $m^3-1\equiv 0 \pmod{3^a}$ has three solutions $\leqq 3^a$ for $a\geqq 2$, one for $a=1$. This proves (1) in case $B=3^a$, p^a, or q^b.

The rest follows by an easy induction. Let (1) hold for some B_1 and let $(P, B_1)=1$. If (3) has n_1 solutions for $B=B_1$ and n_2 solutions for $B=P^a$, then (3) has n_1n_2 solutions for $B=B_1P^a$ (H & W, th. 122). If now we let $P=3$, $P=p$, or $P=q$ and take into account that $n_2=1$ if $P=3$, $a<2$ or $P=q$; and

* References are to Hardy and Wright, *The Theory of Numbers*, Oxford, 1938.

that $n_2=3$ if $P=3^a$, $a\geqq 2$ or $P=p$, it follows that $n=n_1$ or $n=3n_1$, respectively. Using the induction assumption for n_1, the validity of (1) follows.

A Measure-preserving Transformation

4482 [1952, 187]. *Proposed by H. S. Shapiro, Chatham, New Jersey.*

Let $n_1<n_2<\cdots$ be any sequence of positive integers (finite or infinite) and let $T_{n_1n_2\cdots}x$, where x $(0<x<1)$ is expanded in dyadic notation, denote a transformation which alters precisely the n_1, n_2, $\cdots$ digits in x. Then $T_{n_1n_2\cdots}$ is measure-preserving on subsets of (0, 1).

Solution by the Proposer. First, for any integer n and any measurable subset A of (0, 1) we have

$$|T_nA| = |A|, \tag{1}$$

where $|A|$ denotes measure of A. Indeed, T_nx translates x by a distance 2^{-n}, to the left or right according as the nth digit of x is odd or even, and (1) follows at once.

Next, for any interval I with dyadic rational end-points (i.e., having terminating dyadic expansion) and any $T=T_{n_1n_2\cdots}$,

$$|TI| = |I|. \tag{2}$$

Indeed, for k large enough $T_{n_{k+1}n_{k+2}\cdots}I=I$, hence

$$|T_{n_1n_2\cdots}I| = |T_{n_1\cdots n_k}I| = |I|.$$

This last statement follows from the fact that $T_{n_1\cdots n_k}=T_{n_1}\cdots T_{n_k}$, and from (1).

Finally, let A be an arbitrary measurable set; cover it with intervals $I_1, I_2, \cdots$ each of which has dyadic rational endpoints, and such that $\sum|I_n|<|A|+\epsilon$. Write $T_{n_1n_2\cdots}=T$. Now $A\subseteq\sum I_n$ whence $TA\subseteq T\sum I_n\subseteq\sum TI_n$. Therefore

$$|TA| \leqq |\sum TI_n| \leqq \sum|TI_n| = \sum|I_n| < |A|+\epsilon.$$

Since ϵ is arbitrary, $|TA|\leqq A$. Applying the same reasoning with A replaced by TA. and noting $TTA=A$, we have $|A|\leqq|TA|$, and so $|A|=|TA|$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of the Association.

Algebraic Projective Geometry. By J. G. Semple and G. T. Kneebone. Oxford, Clarendon, 1952. 8+404 pages. $7.00.

The authors state that their "main purpose in this book is to construct and develop a systematic theory of projective geometry, and in order to make the system both rigorous and easily comprehensible we have chosen to build it on a purely algebraic foundation." The book is divided into two parts. Part I contains 39 pages and consists of two chapters of an historical and introductory character. Early in Chapter I the following statement is made: "The present chapter is devoted to a rather general consideration of the nature of mathematics and more specifically, of geometry, while Chapter II contains an outline of the intuitive treatment of projective geometry from which the axiomatic theory has gradually been disentangled by progressive abstraction." This gives a good picture of the contents of these two chapters. They are well written and make interesting reading. Chapter II however cannot be fully appreciated by the beginning student since it is just a summary.

The formal development of abstract projective geometry starts with Part II, Chapter III. Throughout, a knowledge of the elements of linear algebra and the calculus of matrices is assumed. Chapter III is devoted to the projective geometry of one dimension. It starts by giving a general definition of projective space of n dimensions over an arbitrary ground field K to be assumed throughout the remainder of the book. However after this definition the base field is assumed to be the field of complex numbers, unless stated to the contrary, and the projective geometry of one, two and three dimensions follows. Frequently they consider real projective geometry and also real projective geometry with "ideal points" adjoined whose coordinates are complex numbers. For example, they start with this "real projective plane with complex points" adjoined to it to derive their affine plane. This is done by singling out a real line to be called "the line at infinity" and considering it as "removed from the plane in the sense that its points will be treated not as actual points of the plane but as ideal points." This gives two types of ideal points to be dealt with in the real affine plane, which then is further specialized to get the Euclidean plane. The same procedure is used in projective geometry of three dimensions. No attempt is made to develop the non-Euclidean geometry, though there are some remarks about it at the end of each of the chapters on projective geometry of one, two, and three dimensions.

The chapter headings after III are: IV. Projective Geometry of Two Dimensions, V. Conic Loci and Conic Envelopes, VI. Further Properties of Conics, VII. Linear Systems of Conics, VIII. Higher Correspondences, Apolarity, and the Theory of Invariants, IX. Transformations of the Plane, X. Projective Geometry of Three Dimensions, XI. The Quadric, XII. The Twisted Cubic Curve and Cubic Surfaces, XIII. Linear Systems of Quadrics, XIV. Linear Transformations of Space, XV. Line Geometry, XVI. Projective Geometry of n Dimensions.

There is a wealth of excellent material in this text book. At the end of each chapter there is a large list of both illustrative and supplementary exercises. This text is a definite contribution to the field of algebraic projective geometry of one, two and three dimensions and lays a good foundation for further development. The text does not attain the standard of rigor sought by some of our modern writers in the field of algebraic geometry. For example, in their definition of a projective space of n dimensions they state that "The projective properties of $S_n(K)$ are those properties of which the expression in every allowable coordinate system is the same." It is not clear from this definition what is meant by the word "expression." Also on page 105 some explanation of what is meant by a "general point" or a "general figure" is offered but even so a lot of detail is left to the reader. However "in general" the authors appear to have accomplished their main purpose in writing this text.

W. F. ATCHISON
University of Illinois

Intermediate College Mechanics. By D. E. Christie, McGraw-Hill Book Co. Inc., New York, 1952. 16+454 pages. $7.00.

Chapters 1–11 of this book contain an excellent treatment of the mechanics of a particle, of a system of particles, and of a rigid body. The remaining six chapters contain brief introductions to a number of independent topics of a more advanced nature. The treatment is three-dimensional, and extensive use is made of vector methods. The arrangement of topics is quite good; explanations are clear, systematic, and detailed; and the text contains many examples worked out carefully and completely. The problem exercises seem quite adequate.

This book is an excellent one for classroom use. The only prerequisite seems to be calculus, though a fuller appreciation of portions of the last six chapters would be realized if the reader has some knowledge of differential equations and advanced calculus.

Chapters 1 and 2 deal with vector algebra. There is some emphasis on geometric proofs. The differentiation and integration of vectors is deferred until Chapter 5. Chapters 3 and 4 contain Newton's laws, and a treatment of the properties of force systems. Also, there appears here the statics of a particle and of a rigid body, including the funicular polygon and the statics of beams. Chapters 5 and 6 deal with the kinematics of a particle and of a rigid body. Chapters 7 and 8 deal with the dynamics of rigid body. The motion of a particle

relative to a moving frame of reference is included, but the motion of a rigid body relative to a moving frame of reference appears only as one of a set of problems at the end of Chapter 11. Chapters 9 and 10 contain some simple methods of solving special problems. Included here are the principles of work-energy, conservation of energy, and virtual work. Chapter 11 deals with impulses and problems involving impacts.

In Chapter 12 there is a treatment of the simple one-dimensional oscillator both with and without damping and an exciting force. Chapter 13 deals with conservative fields of force, and includes a discussion of planetary motions and the two-body problem. Chapters 14 and 15 contain very brief introductions to some of the concepts and equations of the mathematical theories of elasticity and hydrodynamics. Included here are the notions of divergence and curl of a vector, as well as the theorems of Green and Stokes. In Chapter 16, wave motion in one dimension is considered. The partial differential equation governing three elementary vibrating systems is deduced in each case. The general solution of this equation is assumed, and various phenomena such as beats and the Doppler effect are discussed. Chapter 17 contains an introduction to the kinetic theory of a monatomic gas, in which each molecule is treated as a perfectly elastic sphere.

The reviewer can make only one criticism of this book. In his opinion, a more extensive consideration of the motion of a rigid body relative to a moving frame of reference, with applications to the top and gyroscope, would round out nicely an otherwise rather complete book.

G. E. HAY
University of Michigan

Methods of Applied Mathematics. By F. B. Hildebrand, Prentice-Hall, Inc., New York, 1952, 523 pages. $7.75.

"The principal aim of this volume is to place at the disposal of the engineer or physicist the basis of an intelligent working knowledge of a number of facts and techniques relevant to four fields of mathematics which usually are not treated in courses of the 'Advanced Calculus' type, but which are useful in various fields of application."

In the opinion of the reviewer, Professor Hildebrand has achieved his noteworthy aim. His book should be a most useful text and reference volume for seniors and first year graduate students.

It would appear that we are now entering a second phase of American higher education in which greater efforts are being spent on raising the level of instruction in the sciences (including engineering). Professor Hildebrand's book certainly is a large step forward in this direction through his treatment of the four basic topics: Chapter 1, Matrices, Determinants and Linear Equations; Chapter 2, Calculus of Variations and Applications; Chapter 3, Difference Equations; Chapter 4, Integral Equations.

Throughout the text complete derivations of mathematical equations from

physical principles are followed by detailed analyses of the mathematical and the numerical methods that are appropriate. In this way a well rounded and thorough understanding of the basic techniques is transmitted to the reader. Many of the exercises (which are large in number, excellent in quality and supplied with answers or hints) were devised to extend the theory. Of course, Professor Hildebrand's book was not intended as a treatise and it therefore contains careful statements, discussions and references for those topics which could not be completely treated.

Chapter 1 "deals principally with linear algebraic equations, quadratic and Hermitian forms, and operations with vectors and matrices, with special emphasis on the concept of characteristic values. A brief summary of corresponding results in function space is included." The author stops short of deriving the Jordan canonical form for matrices, but describes it with care. His rule for evaluating a determinant is a fine idea for classroom discussion, but should probably not be given to the exclusion of the customary one involving subscripts.

Chapter 2 carefully introduces the variational notation and the Euler equations. Natural boundary conditions, generalized coordinates, Hamilton's principle, and Lagrange's equations are then treated. Normal coordinates, minimal principles for elasticity, *etc.*, and some of the direct methods of the calculus of variations complete the scope of the second chapter.

"The third chapter combines the presentation of available methods for solving the simpler types of difference equations with a description of the application of finite-difference methods to the approximate solution of problems governed by partial differential equations and includes consideration of the troublesome problems of convergence and stability." In his efforts to present in integrated form some of the modern developments of numerical analysis, the author has done a great service. (Even though his discussion of stability and convergence is not as clean-cut or inclusive as it can be made today.)

"The concluding chapter deals with the formulation and theory of linear integral equations, and with exact and approximate methods for obtaining their solutions, particular emphasis being placed on the several equivalent interpretations of the relevant Green's function."

The author has succeeded in writing four self-contained fundamental chapters, and has managed in addition to treat many of those aspects which are common to the various fields. This makes his volume a remarkably well coordinated unit.

EUGENE ISAACSON
New York University

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

MAGIC CIRCLES

S. W. McINNIS, University of Florida

This paper has for its purpose the compiling of the sketchy material on magic circles, enlarging that information, and bringing together the various illustrations found in the literature.

A magic circle consists of $R+1$ concentric circles divided into sections by n diameters or N radii; an arbitrary number C is placed in the central circle and a set of consecutive numbers from a to b are placed in the other sections so that the sum along each diameter is constant. We classify magic circles into three classes: simple, compound, and perfect. Illustrations and properties of these different classes follow.

Let S_r be the sum along any radius, S_d be the sum along any diameter, and S_o be the sum in any ring (excluding the central circle). A property common to all magic circles is

$$S_{d_1} = S_{d_2} = \cdots = S_{d_n}. \tag{1}$$

If (1) is the only property a magic circle has, it is called *simple*. Figures 1 and 2 are illustrations of this class.

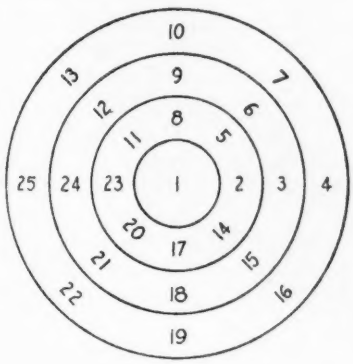

FIG. 1

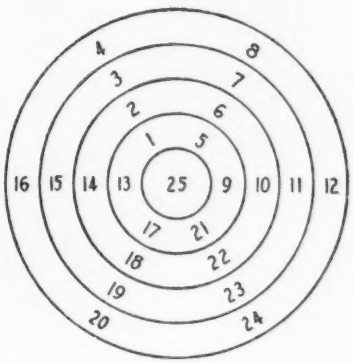

FIG. 2

If simultaneously

$$S_{r_1} = S_{r_2} = \cdots = S_{r_N} \tag{2}$$

and

(3) $$S_{o_1} = S_{o_2} = \cdots = S_{o_R},$$

the magic circle is *compound*. In this event, it follows easily from (2) and (3) that

(4) $$S_r = \tfrac{1}{2}(a + b)R + C,$$

and

(5) $$S_o = \frac{N}{R}(S_r - C).$$

Figures 3 and 4 are examples of this class. If $N = R$, the magic circle is *perfect*. From (5) we have $S_o = S_r - C$. Figure 5 is an example of a perfect magic circle.

Figure 3 is an example of a compound magic circle formed from the numbers 1 to 320 inclusive. Its designer writes:*

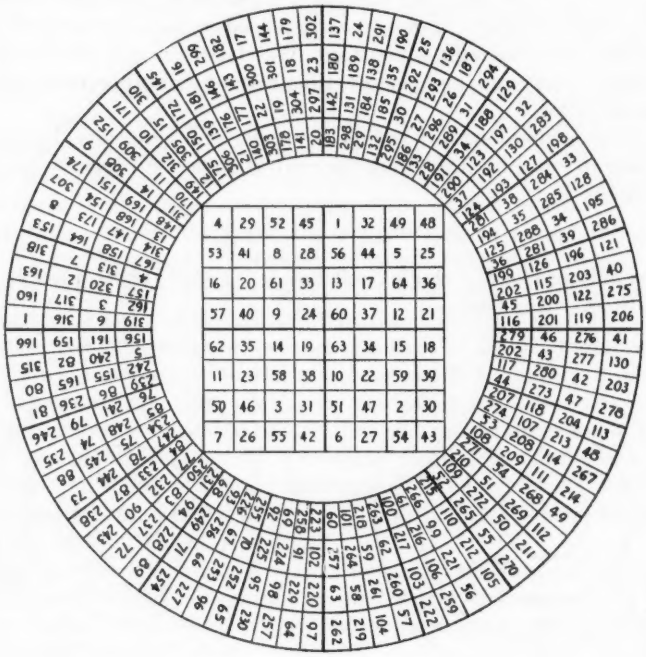

FIG. 3

"The rings of this circular diagram contain the first 320 numbers arranged in twenty subdivisions each of which forms a magic square. Each row, column and diagonal totals 642.

"In the entire ring every right and left diagonal totals 642. Four extra magic

* R. V. Heath, Scripta Mathematica, vol. 3, 1935, p. 340.

squares are formed where the following numbers appear on the outer edge:

267, 124, 49, 112,
227, 254, 84, 72,
207, 174, 9, 152,
187, 294, 129, 32.

"In the center there is a perfect 8 by 8 magic square composed of the numbers 1–64. Each quarter section forms a magic square. Adding any number in a quarter section with the number in like position in the other sections we always get 130.

"The upper tier of the square forms a pan-diagonal rectangle, so does the lower tier."

Here $a=1$, $b=320$, $R=4$, and $N=80$, whence by (4) and (5), $S_r-C=\frac{1}{2}(1+320)\cdot 4=642$ and $S_o=20(642)=12{,}840$.

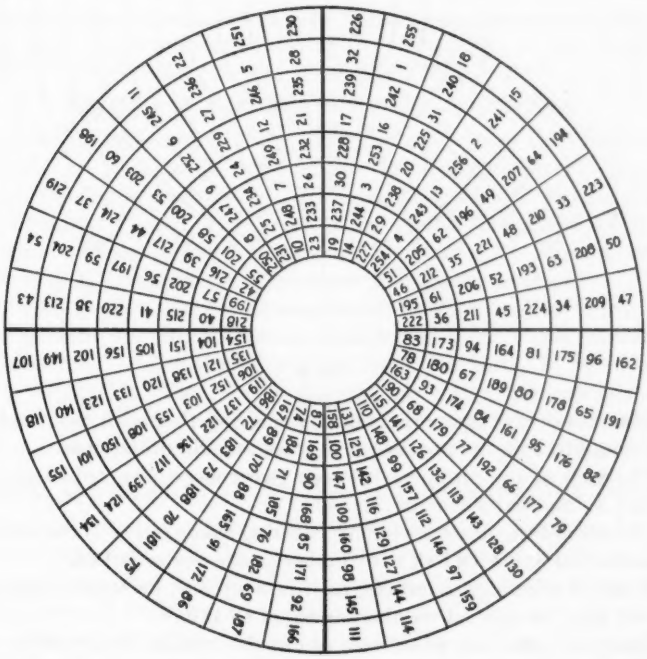

FIG. 4

Figure 4 is another compound magic circle formed from the numbers 1 to 256 inclusive arranged in four subdivisions or quarter sections of the circle such that each quarter forms an 8 by 8 magic square. Here $a=1$, $b=256$, $C=0$, $R=8$, and $N=32$, so that by (4) and (5), $S_r=\frac{1}{2}(1+256)\cdot 8=1028$ and $S_o=4(1028)=4{,}112$.

Figure 5 is an example of a perfect magic circle invented by Dr. Franklin. It is formed by use of the numbers 12 to 75 inclusive with the number 12 repeated in the center circle. The circle consists of nine concentric circles divided into eight parts by eight radii. This circle has the following properties:*

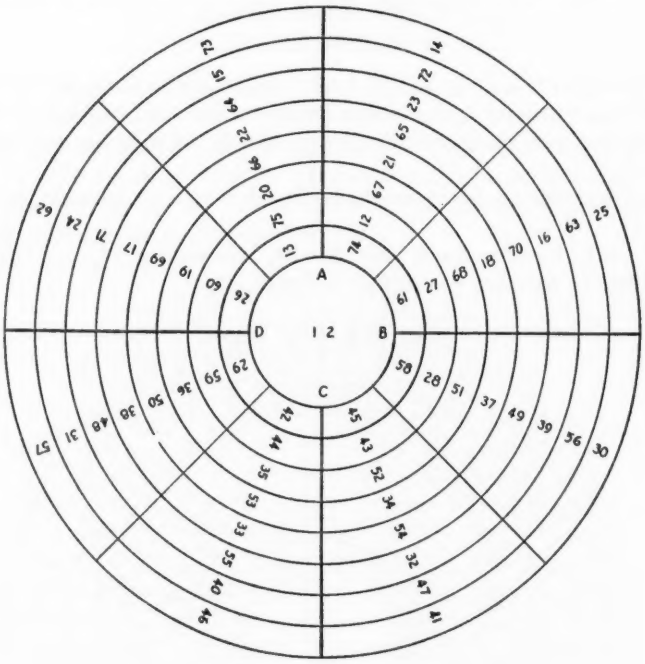

FIG. 5

"1. The sum of all the numbers in any ring together with the number in the middle is equal to 360, the number of degrees in a circumference.

"2. The sum of the numbers between two consecutive radii together with the number at the center is equal to 360.

"3. The sum of the numbers in any half ring taken either above or below the double horizontal line with half the number at the center is 180.

"4. If any 4 adjoining numbers, as if in a square, be taken, their sum, together with half the central numbers is equal to 180.

"5. There are also four other sets of circular spaces, bounded by eccentric circles with regard to the primitive center each of which set contains 5 spaces their centers being at A, B, C, and D." In this case, formulas (4) and (5) give $S_r=\frac{1}{2}(12+75)8+12=360$ and $S_o=360-C$. Thus the sum is 360 also when C is

* Daves and Peck, Mathematical Dictionary and Encyclopedia of Mathematical Science, p 350.

added to each ring.

Two interesting questions arise. Is the number of perfect magic circles finite? If finite, can an example of each different set be illustrated?

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

MEETING OF MATHEMATICS DIVISION OF A.S.E.E.

The Mathematics Division of the American Society for Engineering Education will meet on June 24–26, 1953 at the University of Florida, Gainesville, Florida. In addition to contributed papers the program will include the following: Two lectures by Dr. J. H. Curtiss, Chief of the National Applied Mathematics Laboratories, National Bureau of Standards, on "Programming for Electronic Digital Computers"; a luncheon address by Professor C. O. Oakley, Haverford College, entitled "Mathematics for Engineers Who Will Never Use It"; a symposium on "Instruction in Mathematics for Undergraduate Engineering Students," under the chairmanship of Professor F. W. Kokomoor, University of Florida. For further information about the meeting write to Chairman of the Division, Dr. C. V. Newsom, Associate Commissioner for Higher Education, State University of New York, Albany 1, New York.

CONGRESS AND SEMINAR OF THE CANADIAN MATHEMATICAL CONGRESS

The biennial Seminar of the Canadian Mathematical Congress will be held at Queen's University and the Royal Military College, Kingston, Ontario, August 10 to September 4, 1953. The topics are Topology and Geometry. Research lectures will be given by Professors L. E. J. Brouwer, Henri Cartan, M. H. A. Newman and Beniamino Segre. Seminars will be conducted by Professors H. S. M. Coxeter and H. Zassenhaus. Professors G. deB. Robinson, Peter Scherk, A. W. Tucker and Max Wyman will give instructional lectures.

The Congress will meet from August 31 to September 4 in conjunction with meetings of the Mathematical Association of America and the American Mathematical Society. For further information write to: Canadian Mathematical Congress, Engineering Building, McGill University, Montreal, Quebec, Canada.

MATHEMATICS WORKSHOP AT UNIVERSITY OF ARKANSAS

The University of Arkansas, with the cooperation of the Arkansas Council of Teachers of Mathematics, will conduct a Workshop in Mathematics on July 13–15, 1953. Consultants will be Miss Martha Hildebrandt, Proviso Township High School, Maywood, Illinois, past president of the National Council of Teachers of Mathematics; Professor Maurice L. Hartung, Department of Education, The University of Chicago; Miss Christine Poindexter, Senior High School, Little Rock, Arkansas; Mrs. Virginia Sue Wilson, Supervisor of Elementary Education, College of Education, The University of Arkansas. There will be general meetings and discussion groups for secondary teachers and for elementary teachers.

Participants in the Workshop may obtain accommodations in the facilities of the University. Inquiries regarding the Workshop should be addressed to Professor D. P. Richardson, Department of Mathematics, University of Arkansas, Fayetteville, Arkansas.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1953:

Boston University. June 1 to July 11: Professor Johanson, vector analysis; Professor Scheid, infinite series. July 13 to August 22: Dr. Browder, the theory of games; Professor Noether, introduction to mathematical statistics.

Columbia University. July 6 to August 14: Dr. Brown, introduction to higher algebra; Dr. Finn, differential equations; Professor Cohen, foundations of mathematics, general topology; Professor Levi, theory of functions of a real variable; Professor Lorch, the calculus of finite differences; Professor Murray, probability, Lebesgue measure and integrals; Dr. Taylor, theory of fields, topics in the modern theory of partial differential equations.

Duke University. June 10 to July 18: Professor Carlitz, theory of equations, thesis seminar; Professor Gergen, statistics, thesis seminar; Professor Thomas, plane geometry and trigonometry from the advanced standpoint, thesis seminar. July 21 to August 26: Professor Dressel, probability, thesis seminar; Professor Roberts, thesis seminar. July 20 to July 24: Professor Dressel, Director for Mathematics, Laboratory Conference for Teachers of Science and Mathematics.

Stanford University. June 22 to September 1: Professor Mordell of St. Johns College, Cambridge University, Diophantine equations. July 28 to August 31: Professor Salem of Massachusetts Institute of Technology, trigonometric series and random distribution.

University of Delaware. June 22 to July 31: Professor Barrett, Fourier series; Professor Remage, fundamentals of geometry.

University of Illinois. June 19 to August 15: Professor Day, group theory, functions of real variables.

University of Minnesota, Department of Mathematics, Institute of Technology. June 15 to July 18: Professor Milgram, differential equations and functions of

a complex variable; Professor Munro, intermediate calculus and vector analysis. July 20 to August 22: Professor Polansky, differential equations and advanced calculus.

University of Texas, Department of Mathematics. June 3 to July 15: Professor Ettlinger, differential equations and applications, research in differential equations; Professor Moore, introduction to foundations of geometry, theory of sets; Professor Lane, introduction to the applications of continued fractions, smoothing experimental data; Professors Ettlinger and Moore, thesis for the master's degree, thesis for the degree of Doctor of Philosophy. July 16 to August 26: Mr. Mohat, differential equations and applications, curve fitting; Professor Lubben, topics in modern algebra, introduction to modern projective geometry; Professor Wall, functions of a complex variable, infinite processes.

University of Texas, Department of Applied Mathematics and Astronomy. June 3 to July 15: Professor Craig, vector and tensor analysis, applications of tensor analysis; Professor Greenwood, graphical and numerical computation, mathematical theory of strategy; Mr. Weaver, introduction to the theory of integers; Staff, advanced calculus, differential equations. July 16 to August 26: Professor Cooper, theory of functions of a complex variable, group theory of differential equations; Professor Guy, modern operational methods; Staff, advanced calculus, differential equations.

University of Wisconsin. June 29 to August 21: Professor McCoy of Smith College, determinants and matrices, rings and fields; Professor MacDuffee, survey of the foundations of algebra, tensor analysis; Professor Kleene, elementary plane topology; Professor Fullerton, Fourier analysis; Professor Korevaar, mathematical applications; Dr. Immel, introduction to the theory of probability.

University of Wyoming. June 15 to July 17: Professor Barr, seminar in geometry; Professor S. R. Smith, ordinary differential equations; Professor Steen, college geometry; Professor Varineau, theory of equations, theory of numbers, fundamental concepts of mathematics. July 20 to August 21: Professor Neubauer, history of mathematics; Professor Schwid, partial differential equations; Professor W. N. Smith, mathematical theory of probability.

PERSONAL ITEMS

Professor O. J. Ramler of the Catholic University of America was appointed to represent the Association at the inauguration of President H. R. Anderson of the American University on February 24, 1953.

Dr. Warren Weaver, director for the natural sciences of the Rockefeller Foundation, is President-elect of the American Association for the Advancement of Science; Professor Tibor Rado of Ohio State University has been elected Vice-President and Chairman of the Mathematics Section of the Association; Dean M. H. Ingraham of the University of Wisconsin has been re-elected a member of the Board of Directors for the period 1953–56.

Brown University announces the formation of a physical sciences council

under the chairmanship of Professor William Prager; Professor E. H. Lee replaces Professor Prager as chairman of the Graduate Division of Applied Mathematics.

Iona College reports the following: Mr. F. J. Lane, Brother E. I. Duggan, and Brother S. A. Ryan have been appointed to instructorships; Brother J. G. McKenna has been promoted to the position of Dean of the College; Brother C. A. Lynam has retired.

New York University announces the formation of an Institute of Mathematical Sciences with Professor Richard Courant as scientific director; the Institute consists of three divisions: Mathematics and Mechanics, Mathematics Research, and Computing Services.

Oregon State College announces the following: Professor A. T. Lonseth was granted the Carter Award for inspiring teaching by the School of Science during the fall term of 1952; Mr. R. V. Jamison has been appointed to an instructorship.

Tennessee Polytechnic Institute announces the following appointments: Mr. J. H. Hoelzer of Williams College to an associate professorship; Mr. F. J. Witt, previously head of the Department of Mathematics of Roanoke Rapids High School, North Carolina, to an assistant professorship.

University of Denver makes the following announcements: Assistant Professor Mary E. Waller of Central State College, Edmund, Oklahoma, has been appointed to an instructorship to replace Assistant Professor T. J. Bartlett, who is on leave of absence and is engaged in graduate study; Assistant Professor Olna H. Fant has retired with the title of Professor Emeritus.

At the University of Detroit: Assistant Professor E. D. McCarthy has been promoted to an associate professorship; Instructor E. M. Steinbach has been promoted to an assistant professorship; Mr. S. F. Dice, formerly instructor in mathematics and physics at West Liberty State College, and Mr. J. G. Sowul, previously analytical engineer, Dynamic Analysis Group, Chance Vought Aircraft, Dallas, Texas, have been appointed to instructorships.

University of Florida announces: Mr. H. D. Sprinkle has been appointed to an interim instructorship; Instructor E. J. Lytle, Jr. has returned to the University after two years of military service; Assistant Professor M. E. McCarty has retired.

University of Massachusetts reports the following: Mr. Oscar Litoff of Illinois Institute of Technology has been appointed to an assistant professorship; Mr. A. G. Davis, formerly a mathematics consultant at the Naval Research Laboratory, Mr. Valdemars Punga, and Mr. N. T. Watson have been appointed to instructorships; Mr. Edward Halpern and Mr. W. E. Mientka are on leave of absence during 1952–53.

Institute of Technology, University of Minnesota announces: Professor A. N. Milgram of New York University has been appointed Associate Professor of Mathematics; Associate Professor H. L. Turrittin has been promoted to a professorship.

University of Rochester announces the following: Dr. Walter Rudin, formerly C. L. E. Moore instructor at Massachusetts Institute of Technology, has been appointed to an assistant professorship; Mr. Hewitt Kenyon, previously part-time instructor at the University of California at Berkeley, and Mr. Ralph Raimi, formerly part-time instructor at the University of Michigan, have been appointed to instructorships; Mr. D. C. Barton and Mr. A. E. Danese have been promoted to instructorships; Professor Wladimir Seidel is on sabbatical leave and is at the Institute for Advanced Study.

Reverend H. B. Albiser, formerly at St. Michael's College, Vermont, is teaching now at Cardinal Mindszenty High School, Dunkirk, New York.

Dr. Mabel S. Barnes of Occidental College has been promoted to an assistant professorship.

Mr. D. Y. Barrer of Northwestern University has accepted a position as a staff member with the Operations Evaluation Group, Office of the Chief of Naval Operations, Washington, D. C.

Associate Professor Iacopo Barsotti of the University of Pittsburgh has been promoted to a professorship.

Assistant Professor J. K. Baumgart of Elmhurst College is now in military service.

Mr. F. S. Beckman, formerly an instructor at Pratt Institute, has a position as Senior Mathematician with the International Business Machines Corporation, New York City.

Mr. R. J. Beeber, previously a graduate student at Columbia University, has been appointed to an instructorship at St. Peter's College.

Mr. B. E. Beeman of Texas Agricultural and Mechanical College has been appointed to an instructorship at North Texas State College.

Mr. Joseph Blum, previously of the Armed Forces Security Agency, Washington, D. C., is now an analyst with the Defense Department.

Dr. S. G. Bourne of the University of Connecticut has been appointed to an instructorship at Temple University.

Mr. W. G. Brady, previously a graduate assistant at the University of Pittsburgh, has been appointed to an assistant professorship at Washington and Jefferson College.

Associate Professor J. R. Britton of the University of Colorado has been promoted to a professorship.

Dr. A. H. Brown has been appointed to an instructorship at Rice Institute.

Mr. G. H. Butcher of Howard University has been promoted to an assistant professorship.

Dr. H. H. Campaigne, formerly a mathematician in the Navy Department, Washington, D. C., has been appointed Chief, Armed Forces Security Agency, Defense Department, Washington, D. C.

Assistant Professor W. B. Caton of DePaul University has been promoted to an associate professorship.

Dr. T. Y. Chow, who has been a teaching fellow at Cornell University, has

been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Miss Helen E. Clarkson of Creighton University is now with the Equitable Life Assurance Society of the United States, New York City.

Dr. Mary D. Clement has been appointed Lecturer at the University of Chicago.

Dr. A. H. Clifford has been appointed to an associate professorship at Johns Hopkins University.

Mr. T. S. Dean, formerly president of Design Associates, Sherman, Texas, has a position as Designer-Surveyor with Dean and Alexander, Dallas, Texas.

Assistant Professor R. D. Depew is on leave of absence from Florence State Teachers College, Alabama, and is engaged in graduate study at Vanderbilt University.

Associate Professor R. P. Dilworth of California Institute of Technology has been promoted to a professorship.

Miss Flora Dinkines of the University of Chicago has been appointed to an assistant professorship at the University of Illinois, Chicago, Illinois.

Dr. Mary P. Dolciani of Vassar College has been promoted to an assistant professorship.

Assistant Professor Jim Douglas, Jr. of the University of Alabama has accepted a position as an assistant research engineer with Humble Oil and Refining Company, Houston, Texas.

Dr. Joanne Elliott of Swarthmore College has been appointed to an assistant professorship at Mount Holyoke College.

Mr. Norbert Ellmann of Marquette University is now with the Bell Aircraft Corporation, Niagara Falls, New York.

Dr. Paul Erdös, formerly a visiting lecturer of the University of Aberdeen, Scotland, has accepted a position as a mathematician at the Institute for Numerical Analysis, National Bureau of Standards, Los Angeles, California.

Assistant Professor Trevor Evans is on leave of absence from Emory University and is spending the year at the Institute for Advanced Study.

Mr. J. E. Faulkner, previously a graduate assistant at Kansas State College, has been appointed to an instructorship at Utah State Agricultural College.

Mr. W. R. Ferrante has been appointed to an instructorship at Lafayette College.

Professor A. H. Fox of Union College is on leave of absence and has been appointed a consultant at the Oak Ridge National Laboratory.

Miss Joyce B. Friedman has a position as a mathematician with the Defense Department, Washington, D. C.

Dr. J. W. Gaddum, formerly with the National Bureau of Standards, Los Angeles, California, has accepted a position as a mathematician with the United States Air Force, Washington, D. C.

Dr. L. D. Gates, Jr. of Iowa State College of Agriculture and Mechanic Arts has a position as a mathematician with the Department of Defense, Washington, D. C.

Professor F. C. Gentry of Arizona State College, Tempe, has been appointed to an associate professorship at the University of New Mexico.

Assistant Professor J. B. Giever, formerly of Boston University, is now at the Instrumentation Laboratory, Massachusetts Institute of Technology.

Mr. H. W. Godderz, formerly with the Cessna Aircraft Company, Wichita, Kansas, has been appointed to an assistant professorship at Ohio Northern University.

Mr. M. L. Goldwater is an electronics engineer with Hughes Aircraft Company, Culver City, California.

Mr. G. B. Hedrick of Stanford University has a position as a training specialist with Northrop Aircraft, Hawthorne, California.

Assistant Professor D. M. Hester of Baker University is teaching now at Liberty High School, Texas.

Mr. Joseph Hilsenrath of National Bureau of Standards has been promoted to the position of Physical Science Administrator.

Dr. M. P. Jarnagin of the University of Maryland has accepted a position as a mathematician with the Naval Aviation Ordnance Test Station, Chincoteague, Virginia.

Mr. P. W. M. John, formerly of the University of Oklahoma, is teaching now at Casady School, Oklahoma City, Oklahoma.

Dr. V. L. Klee, who has been at the Institute for Advanced Study, has been appointed to an assistant professorship at the University of Virginia.

Mr. Mark Leum of the University of Iowa has been appointed a mathematician at the Woodward Governor Company, Rockford, Illinois.

Dr. Werner Leutert of the Ballistic Research Laboratories, Aberdeen Proving Ground, has been promoted to Chief, Computing Laboratory.

Mr. R. D. Lowe, previously a graduate assistant at Northwestern University, has accepted a position as a mathematician at International Business Machines, Chicago, Illinois.

Dr. G. H. Lundberg of Vanderbilt University has been appointed to an associate professorship.

Professor A. C. Maddox has retired from Northern State College of Louisiana and is now a member of the staff of Southern State College, Magnolia, Arkansas.

Dr. Jean Maranda has been appointed to an assistant professorship at the University of Montreal.

Assistant Professor William Marcaccio of Xavier University has been promoted to an associate professorship.

Mr. B. W. Marks is teaching at San Jacinto Junior High School, Midland, Texas.

Captain G. T. McCready is now with the Navy Department, Washington, D. C.

Mr. J. E. McKeehan, formerly head of the Department of Mathematics of Skagit Valley Junior College, Mt. Vernon, Washington, is teaching at Lake

Washington High School, Kirkland, Washington.

Assistant Professor K. F. McLaughlin of the United States Naval Academy has been appointed to the position of Associate Professor and Technical Assistant in the Test Service Bureau at Florida State University.

Mr. R. L. McNeal has retired from his position at General Motors Proving Ground, Milford, Michigan.

Professor Karl Menger of Illinois Institute of Technology conducted two programs during March, 1953, in the Institute's new educational TV series.

Dr. W. M. Miller has been appointed to an assistant professorship at Washington and Lee University.

Mr. P. D. Minton, formerly a graduate student at the University of North Carolina, has been appointed to an assistant professorship at Southern Methodist University.

Dr. E. R. Mullins, Jr. has been appointed to an instructorship at Swarthmore College.

Dr. Rufus Oldenburger, chief mathematician of the Woodward Governor Company, Rockford, Illinois, gave a series of lectures recently at the University of Paris on mathematical engineering analysis.

Dr. Daniel Orloff, previously at Bell Aircraft Corporation, Niagara Falls, New York, has accepted a position as a mathematician at Cornell Aeronautical Laboratories, Buffalo, New York.

Associate Professor Margaret Owchar of Southwest Missouri State College has a position as a research worker and statistician with the Manitoba Cancer Relief and Research Institute, Winnipeg, Manitoba, Canada.

Mr. D. H. Pilgrim, who has been a graduate student at State University of Iowa, has accepted a position as a statistician with the United States Rubber Company, Passaic, New Jersey.

Mr. Costas Plithides, previously a graduate student at Columbia University, has been appointed to an instructorship at Newark College of Engineering.

Mr. H. E. Reinhardt, who has been a statistician at the General Electric Company, Richland, Washington, has been appointed to an instructorship at State College of Washington.

Dr. Helene Reschovsky of the University of Connecticut has been promoted to an assistant professorship.

Mr. E. C. Rice, formerly a graduate student at George Peabody College for Teachers, has been appointed to an assistant professorship at Monticello Agricultural and Mechanical College.

Assistant Professor L. G. Riggs of Ohio University has been appointed to an assistant professorship at San Diego State College.

Dr. L. V. Robinson has accepted a position as a mathematician with Wright-Patterson Air Force Base, Dayton, Ohio.

Dr. R. M. Robinson of Iowa State College of Agriculture and Mechanic Arts has been appointed Director of Placement at the University of Arizona.

Dr. Charles Roth of the United States Military Academy has accepted a

position as a guidance counselor at City College of New York City.

Mr. J. A. Ryan of Gonzaga University has accepted a position with Lockheed Aircraft Company, Burbank, California.

Mr. Yomei Sawanobori of Princeton University has a position as a junior mathematician with the Cornell Aeronautical Laboratory, Buffalo, New York.

Mr. R. E. Shear, who has been an assistant at Harpur College, has a position as a mathematician at Aberdeen Proving Ground.

Mr. Milton Siegel, previously at the United States Naval Proving Ground, Dahlgren, Virginia, has accepted a position as a mathematical statistician with the Bureau of the Census, Suitland, Maryland.

Professor Jack Silber of Roosevelt College has returned from a tour of duty as an operations analyst with the Fifth Air Force in Korea.

Mr. D. W. Stoddard of Utah State Agricultural College is now with the Sandia Corporation, Albuquerque, New Mexico.

Miss Dorothy J. Stodola has been appointed to an instructorship at Marquette University.

Mr. Irwin Stoner, who was an assistant scientist at Rosemount Research Center, University of Minnesota, has a position now as a dynamics engineer at Bell Aircraft Corporation, Niagara Falls, New York.

Mr. J. R. Sullivan of Clemson Agricultural College has been promoted to an assistant professorship.

Mr. Peter Terwey, Jr., formerly a part-time instructor at the University of North Carolina, has been appointed to an assistant professorship at Davidson College.

Mr. A. E. Ventriglia of Manhattan College has been promoted to an assistant professorship.

Assistant Professor L. F. Walton of University of California at Santa Barbara has been promoted to an associate professorship.

Mr. W. J. Wells, who was a graduate student at the University of Minnesota, has accepted a position as a mathematician with the United Aircraft Corporation, East Hartford, Connecticut.

Mr. C. S. Williams, Jr. has a position as an electronics engineer with Sandia Corporation, Albuquerque, New Mexico.

President Emeritus H. N. Davis of the Stevens Institute of Technology died on November 3, 1952.

Professor M. W. Dehn of Black Mountain College died on June 27, 1952.

Assistant Professor J. D. Newburgh of Tulane University died on January 3, 1953.

Professor Leigh Page of Yale University died on September 14, 1952.

Professor Emeritus W. P. Russell of Pomona College died on January 10, 1953; he was a charter member of the Association.

Professor Emeritus W. S. Schlauch of New York University died on January 27, 1953.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE JANUARY MEETING OF THE SOUTHWESTERN SECTION

The January meeting of the Southwestern Section of the Mathematical Association of America was held at the New Mexico College of Agriculture and Mechanic Arts at State College, New Mexico, on January 2 and 3, 1953. Professor J. H. Butchart, Chairman of the Section, presided at the sessions.

Fifty persons attended the meetings including the following thirty members of the Association:

O. B. Ader, J. W. Beach, A. I. Benson, J. W. Branson, C. E. Buell, J. H. Butchart, J. T. Clausen, Jr., G. A. Culpepper, D. G. Duncan, R. S. Fouch, Gordon Fuller, F. C. Gentry, A. S. Gregory, E. A. Hazlewood, W. P. Heinzman, M. S. Hendrickson, R. C. Hildner, Verba M. Iturralde, Max Kramer, W. W. Mitchell, Jr., H. E. Pickett, Irene Price, E. J. Purcell, W. L. Shepherd, R. B. Stiles, Deonisie Trifan, R. S. Underwood, Earl Walden, D. L. Webb, J. W. T. Youngs.

The following were elected officers for the year 1953: Chairman, Professor M. S. Hendrickson, University of New Mexico; Vice Chairman, Professor D. L. Webb, Arizona University; Lecturer for 1953, Professor Max Kramer, New Mexico College of Agriculture and Mechanic Arts.

An invited address entitled "The Matter of Topology" was given, following the banquet on Friday evening, by Professor J. W. T. Youngs of Indiana University. Dr. Youngs also read a paper during the day sessions.

The following papers were presented during the two day sessions:

1. *New properties of the quadrilateral*, by Professor J. H. Butchart, Arizona State College, Flagstaff, Arizona.

Using primary methods of inversive geometry, the author showed that the circumcenters of the four triangles of any quadrilateral are the vertices of four triangles similar to the given triangles, and that lines through these circumcenters parallel to the corresponding lines of the given quadrilateral form a homothetic figure, the ratio being -1. A theorem of H. C. Gossard to the effect that the Euler lines of the three triangles formed by the Euler line of a given triangle with its sides constitute a triangle congruent to the given one was proven anew.

2. *A construction in* [4] *for a Cremona involution in* [3], by Professor E. J. Purcell, University of Arizona.

3. *Extended analytic geometry applied to simultaneous equations*, by Professor R. S. Underwood, Texas Technological College.

In this paper a coordinate system is considered which provides readily obtainable loci on a plane for equations in three or more variables. Successive positive half-axes, numbered in counterclockwise sequence, are 90° apart, so that the fifth and sixth axes, for example, coincide respectively with the first and second. Since the simple lattice points of plane analytic geometry are retained, the system is well adapted for finding integral as well as real solutions. The discussion deals chiefly with systems of linear equations in general and with sets of quadratic equations in three or four unknowns.

4. *Remarks on surface area*, by Professor J. W. T. Youngs, Indiana University.

5. *Extension of the Markoff theorem on least squares by means of orthogonal polynomials*, by C. A. Bodwell, Holloman Air Force Base, introduced by Dr. Irene Price, Holloman Air Force Base.

It is shown that the extension of the Markoff theorem can be greatly simplified by means of orthogonal polynomials. This development gives a practical method for computing the variance of any linear function of least squares parameters such as position, velocity, and acceleration of an object in space when the trajectory has been defined in terms of least squares parameters. Applications of the theory were given.

6. *A modification of Cholesky's method*, by Professor M. S. Hendrickson, University of New Mexico.

Given a non-singular system of linear equations $AX = C$, Cholesky's method attempts to find simultaneously a lower triangular matrix L and a unit upper triangular matrix T such that $L^{-1}\bar{A} = \bar{T}$, where $\bar{A}$ is the augmented matrix of the system and $\bar{T}$ is T augmented by the column $L^{-1}C$. This paper shows that whenever one of the principal minors of A is singular the method fails, but that in that case a simple rearrangement of the equations in the system will guarantee success with a very slight loss in time and without altering most of the elements of L and T already found.

7. *Preliminary report on a theory of negative sets*, by Professor R. S. Fouch, Arizona State College, Tempe, Arizona, and Harold Sherman, University of Chicago.

In place of the usual membership notation of set theory, a membership function was introduced with $m(x, A) = 1$ or 0 corresponding respectively to $x \in A$ or $x \notin A$. A generalization was made by introducing -1 as a third value of this function and the negative of a set was then introduced by the relation $m(x, -A) = -m(x, A)$. Various generalizations of the ordinary set operations and relations were given and elementary results stated.

8. *On free modular lattices*, by Professor D. G. Duncan, University of Arizona.

There are several types of modular lattices which contain the free modular lattice generated by three elements as a sublattice, but which still have a finite number of elements (the free modular lattice generated by four elements is infinite). These types of lattice were discussed with particular attention to cases where the number of elements and the dimension have been determined.

9. *The inclusion of lattices within finite Post algebras*, by Professor D. L. Webb, University of Arizona.

Examples of lattices in Boolean and Post algebras were given. In particular the results obtainable from using matric representation of the elements of the algebras.

10. *Mapping of the set of* 120 *points of the symmetric substitution group* G_{120} *in four dimensional space upon a second order variety in five dimensional space*, by Professor Earl Walden, New Mexico College of Agriculture and Mechanic Arts.

Let $(x_1, x_2, x_3, x_4, x_5)$ be the homogeneous coordinates of a point in a space of 4 dimensions and let $(y_1, y_2, y_3, y_4, y_5, y_6)$ be the homogeneous coordinates of a point in 5 dimensional space. Then the mapping is done by:

$$y_i = A_iQ_1^5 + B_iQ_1^3Q_2 + C_iQ_1^2Q_3 + D_iQ_1Q_4 + E_iQ_2Q_3 + F_iQ_5 \qquad \text{where } i = 1, 2, 3, 4, 5, 6$$

and where the Q's are elementary functions of x_1, x_2, x_3, x_4, x_5.

11. *Finite projective planes and their associated Latin squares*, by Professor Charles Wexler and Professor L. J. Paige, Arizona State College, Tempe, Arizona. Paper presented by title.

The following papers having to do with teaching problems and organization of courses were presented on Saturday morning, January 3.

12. *The mathematical preparation of college freshmen from New Mexico high schools*, by Professor F. C. Gentry, University of New Mexico.

13. *Some experiments in the organization and administration of freshman mathematics*, by Professor E. A. Hazlewood, Texas Technological College.

14. *Elimination of variables from systems of equations as an aid in unifying freshman mathematics*, by Professor W. L. Shepherd, Texas Western College.

15. *Contemporary psychology and an experiment in the teaching of mathematics*, by Professor R. S. Fouch, Arizona State College, Tempe, Arizona.

16. *Multi-sensory aids in the teaching of mathematics*, by Mrs. Verba M. Iturralde, Bowie High School, El Paso, Texas.

17. *Report of committee on high school cooperation*, by Professor Max Kramer, New Mexico College of Agriculture and Mechanic Arts.

The committee reported that it is planning to hold statewide mathematical contests in New Mexico this year. Official Association awards are to be given. The program was planned with the cooperation of the Mathematics Section of the New Mexico Education Association.

R. L. WESTHAFER, *Secretary*

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirtieth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Millsaps College, Jackson, Mississippi, on February 13 and 14, 1953. Professor F. A. Rickey, Chairman of the Section, presided at the Friday afternoon session. Professor T. L. Reynolds, Mississippi Vice-Chairman, presided at the Saturday morning session.

There were ninety-seven persons present including the following fifty members of the Association:

T. A. Bickerstaff, Elsie T. Church, W. H. Cleveland, G. J. Corley, Margaret R. Davis, M. P. Dossey, W. L. Duren, Jr., D. O. Etter, L. R. Ford, L. M. Garrison, A. L. Gilmore, Jr., A. C. Grimes, J. A. Hardin, R. H. Hopkins, L. H. Kanter, H. T. Karnes, C. G. Killen, R. J. Koch, Margaret M. LaSalle, Mrs. Helen W. Lindley, Z. L. Loflin, J. W. McClimans, Betty McKnight, R. A. Miller, Benjamin Ernest Mitchell, Benjamin Evans Mitchell, S. B. Murray, M. M. Ohmer, Arthur Ollivier, R. L. O'Quinn, W. V. Parker, B. J. Pettis, T. J. Pignani, P. K. Rees, T. L. Reynolds, F. A. Rickey, A. A. Ritchie, D. R. Scholz, H. F. Schroeder, Fariebee P. Self, S. W. Shelton, P. K. Smith, W. H. Spragens, Jr., V. B. Temple, W. B. Temple, B. B. Townsend, G. J. Trammell, Jr., B. O. Van Hook, Eleanor B. Walters, Dale Woods.

The following officers were elected for the coming year: Chairman, Professor M. E. Gillis, Blue Mountain College; Louisiana Vice-Chairman, Professor M. M. Ohmer, Southwestern Louisiana Institute; Mississippi Vice-Chairman, Professor B. O. Van Hook, Mississippi Southern College; Secretary-Treasurer,

Professor Z. L. Loflin, Southwestern Louisiana Institute.

The following motion was passed at the business session: "The meetings of the Section shall consist of those programs and only those programs open to all members of the Mathematical Association of America and the secretary is instructed to send a notice of the Section meetings to all colleges in Louisiana and Mississippi."

The invited speaker for the meeting was Professor W. L. Duren, Jr., of Tulane University. His address on Friday evening was entitled "The Reform of the Mathematical Curriculum," and at the Saturday morning session, "Elementary Scientific Measurement and the Real Numbers."

The following papers were presented:

1. *Transformations obtained from a certain cubic transformation*, by Professor Elsie T. Church, Northwestern State College.

From results obtained in *A Certain Cubic Transformation*,* the three quadratic transformations A, B, C were derived, where

$$A:\ x_1' = x_2x_3 \qquad B:\ x_1' = \omega x_2x_3 \qquad C:\ x_1' = \omega^2 x_2x_3$$
$$x_2' = x_1x_2 \qquad\qquad x_2' = x_1x_2 \qquad\qquad x_2' = x_1x_2$$
$$x_3' = x_3x_1 \qquad\qquad x_3' = x_3x_1 \qquad\qquad x_3' = x_3x_1.$$

The product of any two of these transformations A, B, C, can be reduced to one of the three linear transformations (α), (β), (γ) where

$$(\alpha):\ x_1' = \omega x_1 \qquad (\beta):\ x_1' = \omega^2 x_1 \qquad (\gamma):\ x_1' = x_1$$
$$x_2' = x_2 \qquad\qquad x_2' = x_2 \qquad\qquad x_2' = x_2$$
$$x_3' = x_3 \qquad\qquad x_3' = x_3 \qquad\qquad x_3' = x_3.$$

The six transformations A, B, C, (α), (β), (γ) form a group H of order six and (α), (β), (γ) form a subgroup of H of order three.

2. *A set of cyclicly related functional equations*, by Professor T. J. Pignani, Loyola University.

In a paper, of this same title, published by W. M. Whyburn in *Bulletin of the American Mathematical Society*, December, 1930, the following system was considered:

$$Y_i' = \sum_{k=1}^{m} A_k(x) Y_{i+kh+m}, \qquad i = 1, 2, \cdots n,\ Y_{i+n} \equiv Y_i,$$

where n is a positive integer, n and h are integers or zero, and $A_k(x)$ are L-integrable functions on an interval of definition X. In this work Whyburn obtained explicit solutions for this system of equations.

A more general extension of this system is as follows:

$$(1) \qquad Y_i' = \sum_{k=1}^{m} A_k(x) Y_{bi+kh+m}, \qquad i = 1, 2, \cdots, n,\ Y_{i+n} \equiv Y_i,$$

under the same hypothesis and where b is a positive integer.

* See this MONTHLY, vol. 59, 1952, pp. 314–315.

In particular, consider the case where $(h, n)=1$ and $b=n$. Then (1) becomes

$$Y_i' = \sum_{s=1}^{m} B_s(x) Y_s, \qquad i = 1, 2, \cdots, n. \tag{2}$$

Solutions satisfying (2) are readily obtained, and they are:

$$Y_j = \exp \int_a^x P(v)dv \left[\int_a^x Q(v) \exp - \int_a^v P(t)dt + K \right] \tag{3}$$

where K is an arbitrary constant, $P(v)=\sum_{s=1}^{m} B_s(v)$, $Q(v)=\sum_{s=1, s\neq j}^{m} c_s B_s(v)$, and $1 \leqq j \leqq n$,
(4) $Y_i = Y_j + c_i$, $i \neq j$, where c_i is an arbitrary constant, $i = 1, 2, \cdots, n$.

Direct substitution of (3) and (4) verify that they satisfy (2).

3. *Quaternions as rotations*, by Miss Margaret M. LaSalle, Southwestern Louisiana Institute.

Quaternions with real coefficients are considered as matrices, with left and right multiplication noted. The general rotation

$$(x, y, 0, 0) + (0, 0, z, t)$$

is expressed as a matrix, M, the angles of rotation being θ and ϕ respectively. The product of two quaternion matrices of the same form does not yield general M; however, the product of two, one from left and the other from right multiplication, gives M, provided the coefficients are expressed in terms of sines and cosines of A and B and $|A+B|=\theta$, $|A-B|=\phi$.

By analogy with two- and three-space, the sum of the squares of rows (and columns) is one, the sum of cross products of rows (and columns) is zero and distances are preserved. It is shown that every vector is changed in length only, that the plane is transformed into itself and that M represents a general rotation.

4. *Mathematics and modern philosophy*, by Professor N. B. Fleming, Millsaps College, introduced by the Secretary.

Modern science and modern philosophy really began when the methods and criteria of mathematics were applied to man's experience and also to his investigation of nature. This development began when the number-mysticism of Pythagoras and the mathematical metaphysics of Plato became more empirical for Kepler and Descartes.

5. *Orthic and oblique hyperbolas*, by Professor V. B. Temple, Louisiana College.

Take non-parallel lines L_1 and L_2, not passing through the origin O. Draw two lines through T, one making a chosen angle w with the vertical through S in P_1, the other making w with the horizontal through S in P_2. Each point traces an orthic or oblique hyperbola as OA rotates about O, according as w is or is not 90°; w is the angle of obliquity and is equal to the angle between the asymptotes.

If e and e' are conjugate eccentricities we have

$$e^2 = \frac{2}{1+\cos w}, \quad e'^2 = \frac{2}{1-\cos w}, \quad \text{and} \quad e^2 + e'^2 = e^2 e'^2 = 4 \csc^2 w.$$

In particular if L_1 and L_2 are respectively parallel to the y and x axes and $w=90°$, P_2 becomes a fixed point, and this point with its symmetric point about O constitute a two-point hyperbola.

6. *The group of automorphisms of cyclic groups*, by Miss Allean McKnight, Sunflower Junior College, Mississippi, introduced by the Secretary.

In order to exhibit an automorphism of a cyclic group a correspondence is set up between

generators. Since a cyclic group is generated by any element whose order is relatively prime to the order g of the group, the order of the group of automorphisms is $\phi(g)$, where ϕ denotes Euler's ϕ-function of g.

Two other properties of the group of automorphisms are given by the following theorems: (1) the group of automorphisms of a cyclic group is abelian, and (2) the group of automorphisms of a cyclic group of order p^α is a cyclic group excluding the case $p=2$, $\alpha>2$.

7. *On a theorem of A. Markoff*, by Professor L. H. Kanter, Mississippi State College.

Starting with the Gauss-Jacobi Mechanical Quadrature Theorem, Markoff's theorem on the variation of the zeros of the classical orthogonal polynomials with a parameter is obtained.

8. *A complete orthonormal set of analytic functions*, by Professor D. R. Scholz, Louisiana State University.

Let $L^2(D)$ denote the class of functions $f(z)$ which are regular and single-valued in a bounded domain D and satisfy the norm

$$\iint_D |f(z)|^2 dxdy = 1.$$

There exists in $L^2(D)$ a complete orthonormal set $f_n(z)$ whose first derivatives are also orthogonal over D and complete in the space of functions with single-valued integrals and finite norm.

9. *Set nets*, by Professor B. J. Pettis, Tulane University.

The present paper consists of extensions to set-valued nets of the results of J. L. Kelley (*Duke Math. Jour.*, 1950, pp. 277–283) on point sets. Some amplifications and applications are included.

10. *The reform of the mathematical curriculum*, by Professor W. L. Duren, Jr., Tulane University.

The time appears to be ripe to establish on a national basis a new undergraduate college curriculum in mathematics. It is becoming clear that the establishment of the long overdue new curriculum, which will bring to undergraduate courses the values of modern mathematical research, will be very slow if not impossible so long as it is attempted by individual schools or textbook writers working independently. For there are too many difficulties involving the national organization of mathematics teaching, such as the previous preparation of students and the adjustments to transferred students. However, the intense interest of recent years in this reform has brought about a general understanding of what ought to be in the new curriculum and how it ought to be presented, *if* it were only possible.

Actually, no new college curriculum can be planned except as a part of a new total curriculum beginning down in the elementary schools and reaching into the first years of graduate work, but the college curriculum is probably the best place to begin, provided that the general outlines of a future new curriculum in the lower schools is kept in mind. Also we must keep in mind the great diversity in needs, training and ability which students bring to the class room and the radically changed nature of American secondary school education. Finally it must be realized that the curricular reform is only a beginning. The real reform is to be in the classroom.

11. *Elementary scientific measurement and the real numbers*, by Professor W. L. Duren, Jr., Tulane University,

Psychologists have recently been interested in what they call "non-parametric" methods because their efforts at measurement lead to linearly ordered sets without a "unit" and without an addition. This may be abstracted mathematically as a "*stream*," which is a linearly ordered set

which is endless, dense, and Dedekind complete. The theory of the stream includes much of the basic theory of functions of a real variable. With additional assumptions of uniformity and of order-related algebraic operations the stream specializes into the real numbers but there are intermediate systems which have interesting relationships to elementary scientific measurement. In fact, it is seldom true that measurements of physical quantities have a natural algebra which includes all of the properties of the real numbers as an ordered field.

12. *The Madison symposium on the training of mathematics teachers*, by Professor Z. L. Loflin, Southwestern Louisiana Institute.

A summary and a critical analysis of the sessions and discussion groups of the Madison Symposium was given.

Z. L. LOFLIN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, Canada, August 31–September 1, 1953.

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary:

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May 2, 1953.
ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.
INDIANA, Ball State Teachers College, Muncie, May 2, 1953.
IOWA
KANSAS
KENTUCKY, University of Louisville, May 9, 1953.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Proving Ground, Dahlgren, Virginia, May 2, 1953.
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, St. Olaf College, Northfield, May 9, 1953.
MISSOURI
NEBRASKA, University of Nebraska, Lincoln, May 2, 1953.
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City, October, 1953.
PACIFIC NORTHWEST, Montana State University, Missoula, June 19, 1953.
PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, United States Military Academy, West Point, May 9, 1953.
WISCONSIN, Mount Mary College, Milwaukee, May 2, 1953.

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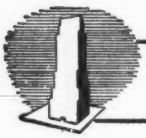

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